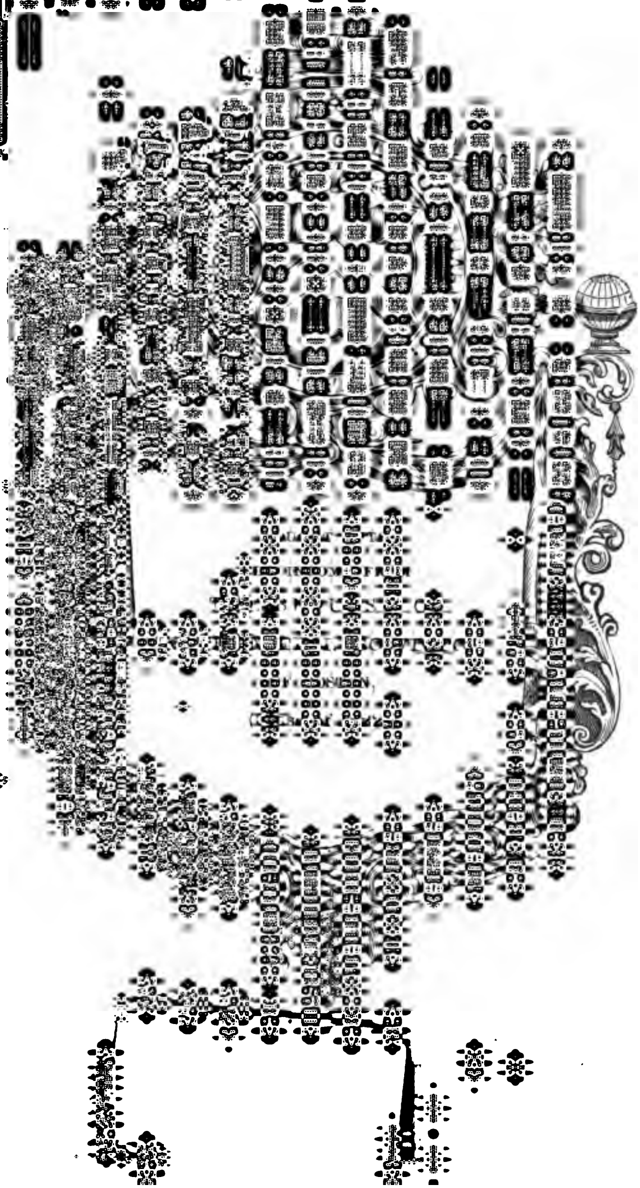

Rivingtons' Mathematical Series

GEOMETRICAL CONIC SECTIONS

AN INTRODUCTION

TO THE STUDY OF

GEOMETRICAL CONIC SECTIONS

BY

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PREFACE.

In writing this book my intention has been to present the Elementary properties of the Conic Sections in a clear and concise form.

I gratefully acknowledge my obligation to a large number of Authors who have written on this subject, and particularly to the Rev. Dr. Taylor, from whose *Geometry of Conic Sections* I have gained much information.

Many of the Exercises given at the end of the book have been taken, with the permission of the Authors, from the Rev. H. Latham's *Geometrical Problems in the Properties of the Conic Sections*, and the Rev. Dr. Morgan's *Collection of Mathematical Problems;* the rest are selections from the *Senate-House Riders*, and from a few recent College Examination Papers.

I offer my thanks to the friends who have improved this work by their advice, and I ask each reader of the book to send me suggestions for its further improvement.

J. HAMBLIN SMITH.

42 Trumpington Street, Cambridge,
December 1886.

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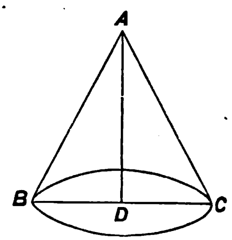
CHAPTER I.

INTRODUCTORY REMARKS.

I.

Conical Surfaces.

1. Euclid defines a Cone as a *solid* figure, described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.



Let ADC be the right-angled triangle, in the plane of the paper. Produce CD to B, making $DB = CD$, and join AB. As the triangle ADC revolves about AD, the side AC describes the *surface* of the cone.

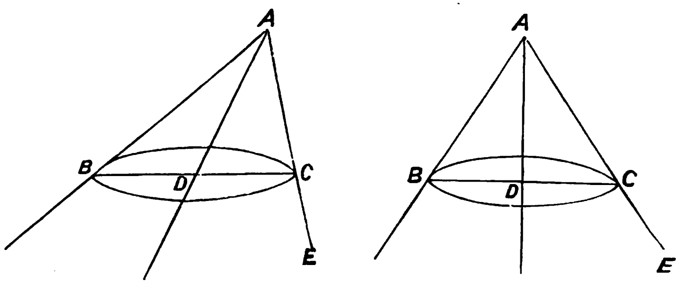
The line AC, revolving from right to left, traces out the surface of the cone *above* the plane of the paper, till AC coincides with AB. Continuing to turn, and having its end C always on the circumference of the circle, perpendicular to the plane of the paper, of which D is the centre and DC the radius, AC as it revolves from left to right traces out the surface of the cone *below* the plane of the paper.

The straight line AD is called the *Axis* of the Cone.

The point A is called the *Vertex* of the Cone.

The circle BC is called the *Base* of the Cone.

2. Next, suppose A to be a fixed point in the plane of the paper, and BC a fixed circle whose plane is inclined at any angle to the plane of the paper, and whose centre D is in the plane of the paper.

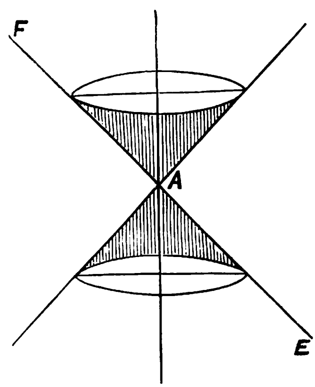


Let a straight line AE, of unlimited length, move so as always to pass through the point A and to meet the circumference of the circle BC; then AE will generate the

surface of a cone. The line AE in any position is called a *Generating Line* of the cone, and the part of AE between A and BC, as AB or AC, is called a *Side* of the cone.

When the axis AD is perpendicular to the plane of the circle BC, the surface generated is called a *Right Circular Cone;* and when the axis is not perpendicular to the plane of the circle, the cone is called *Oblique.*

3. Lastly, if the generating line EAF is unlimited in *both* directions from the vertex, the surface generated is made up of two parts, similar in all respects, on opposite sides of the vertex. Thus, the way in which the complete surface of the Right Circular Cone is generated is shown by the following diagram:

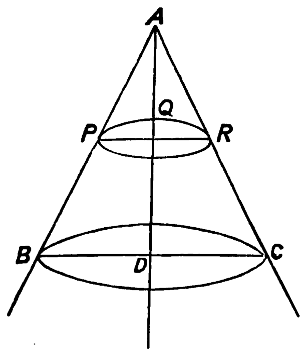


4. The curves obtained by cutting a Conical Surface by a plane are called CONIC SECTIONS.

II.

Sections of a Right Cone by a Plane.

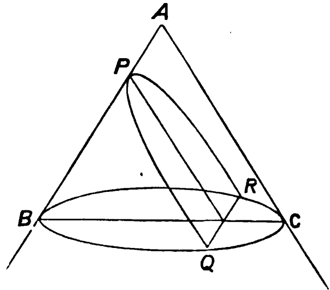
5. If the cutting plane is perpendicular to the axis of the cone, the section is a circle.



Thus if the plane PQR, perpendicular to the axis AD, cuts the cone, the section PQR is a circle.

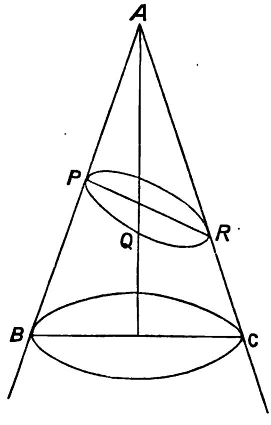
6. The curves to which the name *Conic Sections* is by general usage restricted, are three in number, the *Parabola*, the *Ellipse*, and the *Hyperbola*.

I. If the cutting plane is parallel to a side of the cone, the section is a *Parabola.*



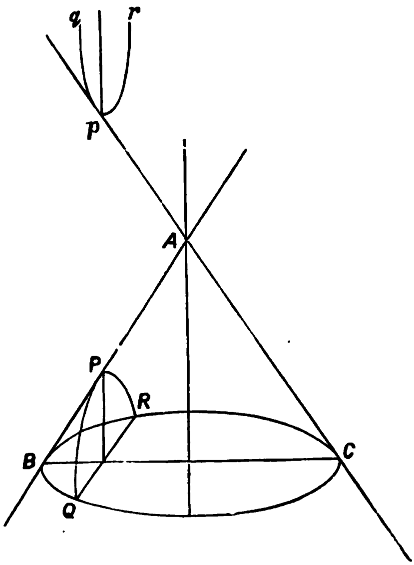
Thus, if the plane *PQR,* parallel to the side *AC,* cuts the cone, the section *QPR* is a parabola.

II. If the cutting plane cuts all the generating lines on the same side of the vertex, the section is an *Ellipse.*



Thus, if the plane *PQR* cuts all the generating lines of the cone *ABC* on the same side of the vertex *A,* the section *PQR* is an *Ellipse.*

III. If the cutting plane meets *both parts* of the cone the section is a *Hyperbola.*



Thus, if the plane *PQR* cuts the lower part of the cone, and the plane being produced cuts the upper part of the cone also, the sections *QPR*, *qpr* are opposite branches of a Hyperbola.

So much is necessary, and sufficient, to explain why these curves are called Conic Sections.

III.

Independent Definition and Treatment of the Conic Sections without reference to the Cone.

7. In the investigation of the properties of the Parabola, Ellipse, and Hyperbola, we do not *define* these curves as sections of a cone. We define them as curves traced out by a point moving continuously in one plane, according to some constant law, and we deduce the properties of each curve from the definition, employing reasoning of the same kind as that which is used in Euclid's Elements. The propositions in Euclid which are of primary importance in the geometry of Conic Sections, are those relating to the similarity of triangles, especially Propositions 2, 3, A, 4, 6, 8 (in my edition, XII.) of the Sixth Book.

8. I shall assume the truth of the following theorems, which are given as exercises in my edition of Euclid :—

(1.) If any two straight lines be cut by three parallel straight lines, they are cut proportionally (Eucl. p. 246).

(2.) The straight line bisecting the angle between the equal sides of an isosceles triangle, bisects the base, and is perpendicular to the base (Eucl. p. 21).

(3.) If two right-angled triangles have the hypotenuse and one side of the one equal respectively to the hypotenuse and one side of the other, the triangles are equal in all respects (Eucl. p. 43).

(4.) The straight lines bisecting the interior and exterior angles between two straight lines are perpendicular to each other (Eucl. p. 41, Ex. 5).

(5.) The square on a straight line is equal to four times the square on half the line (Eucl. p. 79).

(6.) If two straight lines be parallel to two other straight lines, each to each, the first pair make the same angles with one another as the second (Eucl. p. 50).

(7.) If in any quadrilateral the opposite angles be together equal to two right angles, a circle may be described about that quadrilateral (Eucl. p. 153).

(8.) The sum of the squares on any two sides of a triangle is equal to twice the sum of the squares on half the base and on the straight line joining the vertical angle with the middle point of the base (Eucl. p. 91).

9. The use of symbolic representation in the geometry of Conic Sections is generally less restricted than in the proofs of the Elements of Euclid. But though some signs and symbols, borrowed from algebraic notation, are usually employed in this branch of geometry, the processes must always be such as to admit of a geometrical interpretation.

Thus AB^2 stands for the square *on* the straight line AB, and $AB.CD$ stands for the rectangle contained by the straight lines AB and CD.

Also, if AB and CD be two straight lines, we may write

$$(AB+CD)^2=AB^2+CD^2+2AB.CD,$$

provided that we interpret this expression in accordance with Euclid's theorem, Book II. Prop. 4.

And we may also write

$$(AB-CD)^2=AB^2+CD^2-2AB\,.\,CD,$$

provided that we interpret the expression in accordance with Euclid II. 7. We may even write

$$AB^2-CD^2=(AB+CD)(AB-CD),$$

provided that we interpret the expression in accordance with the geometrical theorem that the difference between the squares on two straight lines is equal to the rectangle contained by the sum and the difference of the two lines (Eucl. p. 84).

IV.

Definition of a Conic Section.

10. A *Conic Section* is a curve traced by a point, which moves in such a way that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

The fixed point is called the *Focus;* the fixed straight line is called the *Directrix;* and the constant ratio is called the *Eccentricity.*

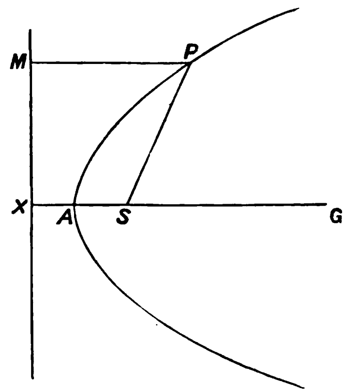
The curve is called a *Parabola,* an *Ellipse,* or a *Hyperbola,* according as the eccentricity is equal to unity, less than unity, or greater than unity.

Note.—The word *Conic* is often used, for the sake of brevity, instead of *Conic Section.*

CHAPTER II.

The Parabola.

11. Def.—If a point P move in such a way that its distance SP from a fixed point S is always equal to its perpendicular distance, PM, from a fixed straight line MX, the curve traced out by P is called a *Parabola*.



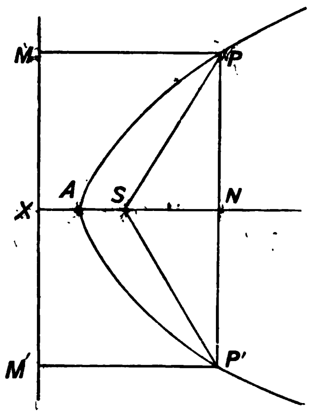
The point S is called the *Focus;* the line MX is called the *Directrix.*

Through S draw XSG perpendicular to the directrix; the line XSG is called the *Axis.*

Bisect SX in A; then, from the Definition, A is a point in the parabola: this point is called the *Vertex.*

SP is called the *Focal Distance* of the point P.

12. *To trace the curve by the determination of successive points, the focus and the directrix being given.*



Through N, any point in XA produced, draw a straight line perpendicular to the axis. With centre S and radius equal to NX, describe a circle cutting the straight line in P, P'. Draw PM, $P'M'$, perpendiculars to the directrix.

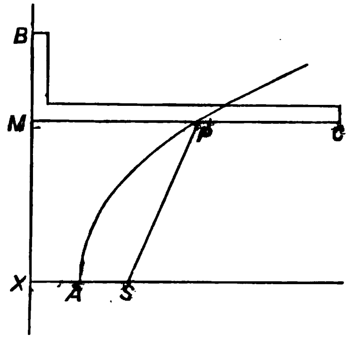
$$\text{Then } SP = NX = PM, \text{ and}$$
$$SP' = NX = P'M';$$
$$\therefore P \text{ and } P' \text{ are points on the curve.}$$

In this way, by taking N at other points in XA produced, we might determine any number of points on the curve.

Since $PN = P'N$, the curve will be symmetrical with regard to the axis.

And since NX may have any value from AS to infinity, it follows that the curve will extend infinitely on each side of the axis.

13. *The Mechanical Construction of a Parabola.*



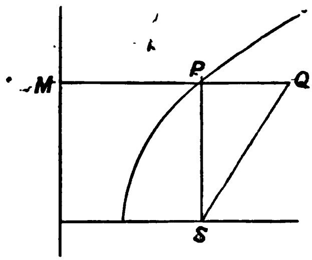
Let BMC, a solid bar shaped like a carpenter's square—that is, having the arms BM, MC at right angles—be made to slide with the end BM always on the directrix BX. Let a string of length MC be fastened at C, and let the other end of the string be fastened to the focus S. As M slides along the directrix let a pencil at P keep the string tight: then P will trace out a portion of a parabola, since $SP+PC=CM$, and therefore $SP=PM$.

14. *The focal distance of a point within the parabola is less than the perpendicular distance of the point from the directrix.*

Let Q be any point within the parabola.

Draw QM perpendicular to the directrix, cutting the curve in P. Join SQ, SP.

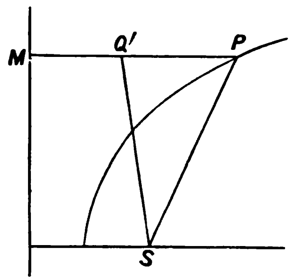
Then $QP+PS$ is greater than QS; (Eucl. I. 20.)



$\therefore$ $QP+PM$ is greater than QS;

$\therefore$ QM is greater than QS; and $\therefore$ QS is less than QM.

15. *The focal distance of a point outside the parabola is greater than the perpendicular distance of the point from the directrix.*



Let Q' be any point outside the parabola.

Draw $Q'M$ perpendicular to the directrix, and let MQ' produced meet the curve in P. Join SQ', SP.

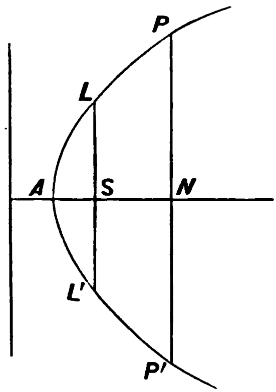
Then $Q'S+Q'P$ is greater than SP; (Eucl. I. 20.)

$\therefore$ $Q'S+Q'P$ is greater than PM;

$\therefore$ $Q'S+Q'P$ is greater than $Q'M+Q'P$;

$\therefore$ $Q'S$ is greater than $Q'M$.

16. *Definitions of* **Ordinate, Abscissa,** *and* **Latus Rectum.**



Draw the chord PNP' perpendicular to the axis.

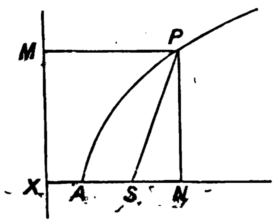
Then PN is called the *Ordinate* of the point P; and AN is called the *Abscissa* of the point P.

It has been shown in the method for the construction of the curve, given in Art. 12, that $P'N = PN$.

PP' is called the *Double Ordinate* of the point P.

The double ordinate LL' through the focus is called the *Latus Rectum.*

17. *If PN be an ordinate to the parabola at the point P, then* $PN^2 = 4AS \, . \, AN$.



For $PN^2 + SN^2 = SP^2$ (Eucl. I. 47.)

$$= MP^2$$

$$= XN^2$$

$$= XS^2 + SN^2 + 2XS \, . \, SN; \text{ (Eucl. II. 4.)}$$

and $\therefore PN^2 = XS^2 + 2XS \, . \, SN$

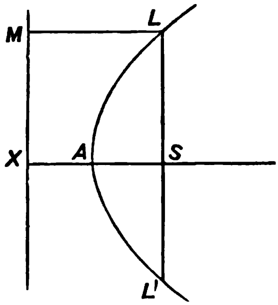
$$= XS \, (XS + 2SN).$$

$$= 2AS \, (2AS + 2SN).$$

$$= 2AS \, . \, 2AN.$$

$$= 4AS \, . \, AN.$$

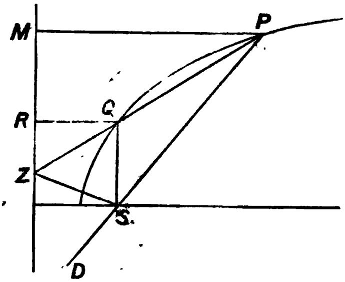
18. *The Latus Rectum* $LL' = 4AS$.



Draw LM perpendicular to the directrix.

Then $LL' = 2LS = 2LM = 2XS = 4AS$.

19. *If the chord* PQ *meet the directrix in* Z, *the line* ZS *will bisect* QSD, *the exterior angle between* PS *and* QS.



Draw PM and QR perpendiculars to the directrix.

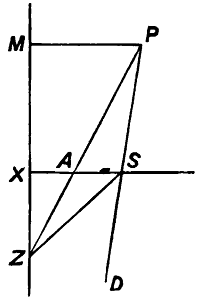
Then $PS : QS = PM : QR$

$= PZ : QZ$. (Eucl. VI. 2.)

$\therefore$ ZS bisects the angle QSD. (Eucl. VI. A.)

This proposition is true for all Conic Sections.

20. From this theorem is derived a method of describing the parabola by the determination of a succession of points when the focus and the directrix are given.



Let MZ be the directrix, and S the focus.

Draw SX perpendicular to MZ, and bisect SX in A. Then A is a point on the curve.

Take any point Z in the directrix; join ZA, ZS, and make $\angle ZSD = \angle ZSA$.

Now SA, being equal to AX, is less than AZ, and $\therefore \angle AZS$ is less than $\angle ASZ$, that is, than $\angle ZSD$. Hence $\angle AZS$ and the angle between ZS and DS produced are together less than two right angles, and $\therefore ZA$, DS will meet when produced. Let them meet in P; then P shall be a point on the curve. Draw PM perpendicular to MZ.

Then, since ZS bisects $\angle ASD$,

$$SP : SA = PZ : AZ \qquad \text{(Eucl. VI. A.)}$$

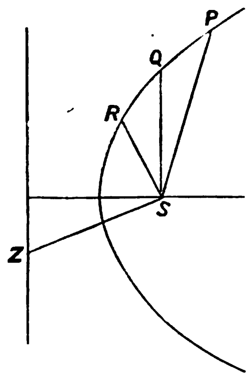
$$= PM : AX; \qquad \text{(Eucl. VI. 2.)}$$

and $\therefore SP = PM$;

$\therefore P$ is a point on the curve.

Note.—A similar method is applicable to the other Conic Sections.

21. *No straight line can meet the parabola at more points than two.*



If it be possible, let P, Q, R be three points in which a straight line meets the parabola, and let Z be the point in which the line meets the directrix.

Join SP, SQ, SR, SZ.

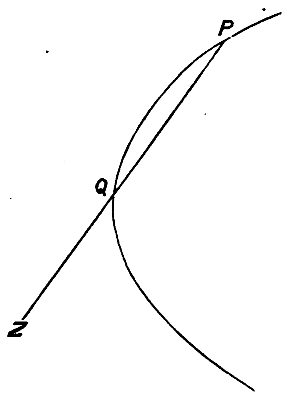
Then ZS bisects the exterior angle between SP and SQ (Art. 19), and ZS also bisects the exterior angle between SP and SR (Art. 19), which is impossible.

Hence a secant to the parabola cuts the curve in two points only.

This proposition is true for all Conic Sections.

The Tangent.

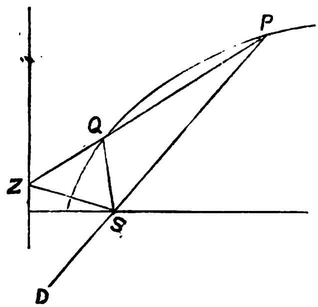
22. A *Tangent* to a parabola is the ultimate position of a secant, when the two points in which the secant cuts the curve have become coincident.

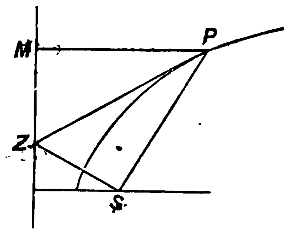


Thus if ZQP be a secant cutting the parabola in Q and P, and if we make ZQP turn about P till Q coincides with P, the secant in its ultimate position will become a tangent at P.

This definition is applicable to all Conic Sections.

23. *If the tangent at P meet the directrix in Z, ZSP is a right angle, and ZP bisects the angle between the focal distance PS and the perpendicular PM on the directrix.*





Let P and Q be two points near each other on the curve.

Draw the chord PQ, and produce it to meet the directrix in Z.

Then (Art. 19) ZS bisects the angle QSD. Now let Q move up to P; then the angle QSD continually increases, and ultimately, when Q coincides with P, and the line QS coincides with PS, QSD becomes equal to two right angles, and therefore each of the equal angles, into which QSD is divided by ZS, becomes a right angle.

In this position ZP is the tangent at P.

Then in the right-angled triangles MPZ, SPZ,

since $MP = SP$, and PZ is common,

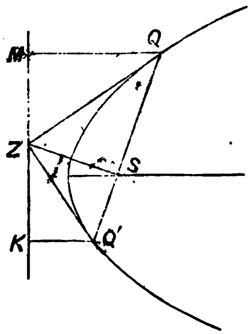
$\therefore$ the triangles are equal in all respects (Eucl. I. E. Cor.),

and $\therefore\ \angle MPZ = \angle SPZ$.

Cor. Hence it follows that the tangent at the vertex is perpendicular to the axis.

Properties of Focal Chords.

24. *Tangents at the ends of a focal chord intersect at right angles on the directrix.*



Let QSQ' be a focal chord.

Draw SZ, meeting the directrix in Z, at right angles to QQ'.

Then ZQ, ZQ' are tangents at Q, Q'. (Art. 23.)

Draw QM, $Q'K$ perpendiculars to the directrix.

Now $\angle MQQ' + \angle KQ'Q =$ two right angles, (Eucl. I. 29)

$\therefore \angle ZQS + \angle ZQ'S =$ a right angle, (Art. 23.)

$\therefore \angle QZQ'$ is a right angle. (Eucl. I. 32.)

Cor. Since $ZM = ZS$, and $ZK = ZS$,

$$\therefore ZM = ZK.$$

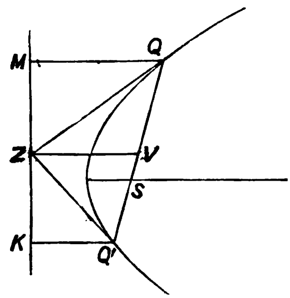
Ex. 1. Show that ZS is the common chord of circles described on ZQ and ZQ' as diameters.

Ex. 2. Show that the angles ZMS, ZQS are equal.

Ex. 3. Show that the triangles ZQS, $Q'ZS$ are similar.

Note.—The three Articles which follow must be regarded as continuations of Art. 24.

25. *If from Z, the point on the directrix in which the tangents at the ends of the focal chord QQ' meet, ZV be drawn parallel to the axis to meet QQ' in V, then $QV = Q'V = ZV$.*



Since ZV is parallel to MQ and KQ',

and since $MZ = KZ$, (Art. 24.)

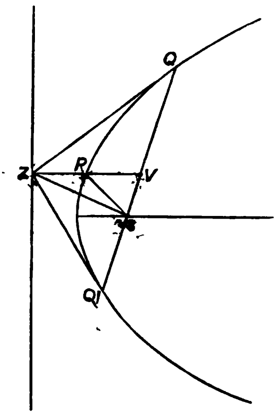
$\therefore QV = Q'V$. (Art. 8 (1).)

Hence, since QZQ' is a right angle, (Art. 24.)

V is centre of the circle described about the triangle QZQ',

and $\therefore QV = Q'V = ZV$.

26. *If from Z, the point on the directrix in which the tangents at the ends of the focal chord QQ' meet, ZV be drawn parallel to the axis cutting the parabola in P, then $QQ' = 4SP$.*



Since ZSV is a right angle, (Art. 23.)

$\therefore$ ZV is a diameter of the circle described about the triangle ZSV.

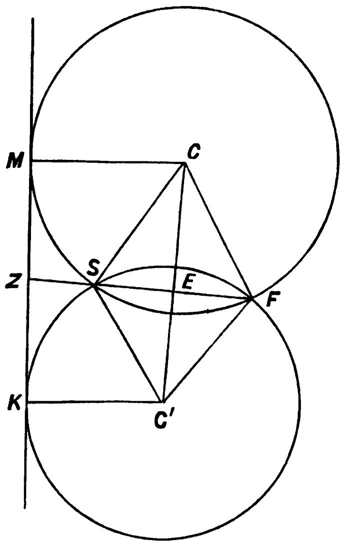
And since $SP = PZ$, from the definition of the curve,

$\therefore$ P is the centre of this circle.

Hence $ZV = 2SP$,

and $\therefore$ $QQ' = 4SP$. (Art. 25.)

27 (Part I.). *If CM, $C'K$ be the perpendiculars to the directrix from the ends of a chord CC', the perpendicular drawn to CC' through the focus will bisect MK.*



Let S be the focus (see diagram on the opposite page). With centre C and radius CS or CM describe a circle MSF. With centre C' and radius $C'S$ or $C'K$ describe a circle KSF. The directrix touches these circles at M, K. (Eucl. III. 16.) Produce the common chord FS to meet the directrix in Z.

Then, since $FZ \cdot ZS = MZ^2$, (Eucl. III. 36.)

and $FZ \cdot ZS = KZ^2$, (Eucl. III. 36.)

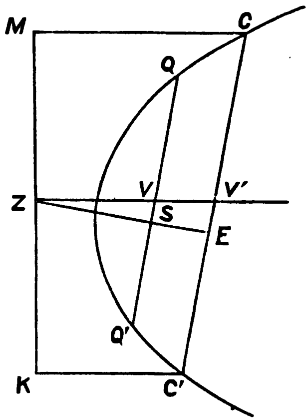
$\therefore MZ = KZ$.

Also, since $CS = CF$, and $C'S = C'F$,

$\therefore \angle SCC' = \angle FCC'$; (Eucl. I. 8.)

and $\therefore CC'$ is perpendicular to SF. (Art. 8 (2).)

27 (Part II.). *If from Z, the point on the directrix where the tangents at the ends of the focal chord QQ' meet, ZV be drawn parallel to the axis, the middle points of all chords parallel to QQ' will lie in the line ZV.*



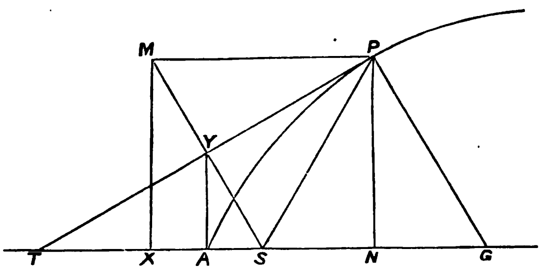
Let CC' be any one of a system of chords parallel to QQ', and let ZV meet CC' in V'. Then we have to prove that $CV' = C'V'$. Draw CM, $C'K$ perpendiculars to the directrix.

Join ZS, and produce ZS to meet CC' in E.
Then ZEC is a right angle, (Art. 24.)
and $\therefore MZ = KZ$. (Part I. of this Art.)
And since ZV' is parallel to MC and KC', and $MZ = KZ$,
$\therefore CV' = C'V'$. (Art. 8 (1).)

Note.—The inclination of the focal chord to the axis is not involved in the foregoing propositions (Arts. 24-27), and hence we conclude that if we have any system of parallel chords in a parabola, the middle point of the focal chord of the system, and consequently the middle points of all the chords, will lie in a line parallel to the axis, which line passes through that point on the directrix in which the tangents at the ends of the focal chord of the system meet.

28. *Properties of the Tangent and the Normal.*

Let PT be a tangent to the parabola at P, meeting the axis in T. Draw PM perpendicular to the directrix, PN perpendicular to the axis, and PG perpendicular to the tangent and meeting the axis in G.



Then PG is called the *Normal* at P;
NG is called the *Subnormal;*
NT is called the *Subtangent.*

Let Y be the intersection of PT and SM.

Then we can prove the following relations:

(1.) Since SPM is an isosceles triangle,
and PY bisects the angle MPS, (Art. 23.)
$\therefore$ PY is perpendicular to SM, (Art. 8 (2).)
and $MY=SY$.

(2.) If we join AY, it bisects SM and SX, and
$\therefore$ AY is parallel to MX, and (Eucl. VI. 2.)
$\therefore$ AY is the tangent at the vertex. (Art. 23.)

(3.) Since angle STP = alternate angle MPT

$= \text{angle } SPT,$ (Art. 23.)

$\therefore SPT$ is an isosceles triangle;

and SY is perpendicular to PT,

$\therefore TY = PY;$

and, since AY is parallel to NP,

$\therefore TA = AN,$ (Eucl. VI. 2.)

and $\therefore NT = 2AN.$

(4.) Since PG is parallel to YS,

and $PY = TY$,

$\therefore SG = ST$ (Eucl. VI. 2.)

$= SP = PM.$

(5.) Since PN is parallel to YA, and $PY = TY$,

$\therefore PN = 2AY.$ (Eucl. VI. 4.)

(6.) Since PNG, YAS are similar triangles,

and $PN = 2AY$,

$\therefore NG = 2AS.$ (Eucl. VI. 4.)

(7.) Since TPG is a right angle, and PN perpendicular to TG,

$\therefore PN^2 = TN \,.\, NG$ (Eucl. VI. 8.)

$= 2AN \,.\, 2AS$

$= 4AS \,.\, AN.$ (See Art. 17.)

(8.) Since SYT is a right angle, and YA perpendicular to TS,

$\therefore SY : AS = ST : SY$ (Eucl. VI. 8.)

$\therefore SY^2 = AS \,.\, ST$

$= AS \,.\, SP.$

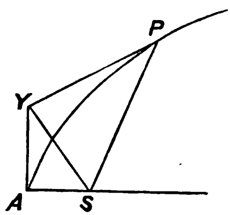
(9.) Since SY bisects the angle ASP,

$\therefore \angle ASY = \angle YSP;$

$\therefore$ since YAS and PYS are right angles, $\angle AYS = \angle YPS.$

29. Results (8) and (9) are to be specially noticed, because they establish for a particular case three important Theorems, which, as we shall prove hereafter, are true for all cases.

Observe then that two tangents YP, YA are drawn to the curve from the same point Y, and



I. $\angle YPS = \angle AYS$.

That is, the angle between the first tangent and the focal distance of the point P, *to* which it is drawn, is equal to the angle between the second tangent and the focal distance of the point Y, *from* which it is drawn.

II. $\angle YSP = \angle ASY$.

That is, the tangents subtend equal angles at the focus.

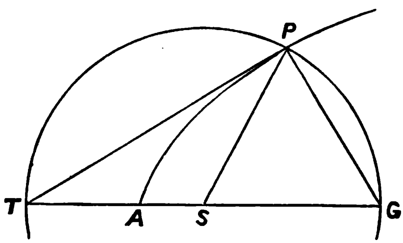
III. $SY^2 = AS \,.\, SP$.

That is, the focal distance of the point *from* which the tangents are drawn is a mean proportional between the focal distances of the points *to* which the tangents are drawn.

Another example of the truth of these Theorems in a particular case will be found in Art. 24, where

$$\angle ZQS = \angle Q'ZS, \text{ and } \angle ZSQ = \angle ZSQ'.$$

30. *To draw a tangent to a parabola from a given point on the curve.*



Let P be the given point.

With the focus as centre, and the focal distance SP as radius, describe a circle cutting the axis in T and G.

Then since $SP = ST = SG$,

$\therefore$ PT is the tangent at P, (Art. 28.)

and PG is the normal at P. (Art. 28.)

...om an external

... centre O and

... x in M and K.

... trix, meeting

... the curve.

... being equal,

(Eucl. I. 8.)

(Art. 23.)

... Q' being equal,

(Art. 23.)

$\alpha+\beta+\gamma = 2$ rt.
$\alpha+\beta+\gamma =$ "

COR. I. *To show that* $\angle OSQ = \angle OSQ'$.

Draw OB parallel to the axis of the parabola, and therefore perpendicular to the directrix, and let it meet the directrix in B.

Then in the right-angled triangles OMB, OKB,

$$\text{since } OM = OK,$$
$$\therefore MB = KB, \qquad \text{(Eucl. I. E. Cor.)}$$
$$\text{and } \angle OMB = \angle OKB.$$
$$\text{Hence } \angle OMQ = \angle OKQ',$$
$$\text{and } \therefore \angle OSQ = \angle OSQ'.$$

COR. II. *To show that* $\angle OQS = \angle Q'OS$
and $\angle OQ'S = \angle QOS$.

The six angles at the point O = four right angles, and these six angles are arranged in pairs of equal angles marked α, β, γ in the diagram.

Hence $2\angle BOM + 2\angle MOQ + 2\angle Q'OS$ = four right angles.

$$\therefore \angle BOM + \angle MOQ + \angle Q'OS = \text{two right angles};$$
$$\text{that is, } \angle BOQ + \angle Q'OS = \text{two right angles}.$$

But $\angle BOQ + \angle OQM$ = two right angles; (Eucl. I. 29.)

$$\therefore Q'OS = \angle OQM = \angle OQS.$$

Similarly we may show that $\angle OQ'S = \angle QOS$.

Also, since OSQ, $Q'SO$ have been thus shown to be similar triangles, it follows that

$$QS : OS = OS : Q'S,$$
$$\text{or } QS \,.\, Q'S = OS^2.$$

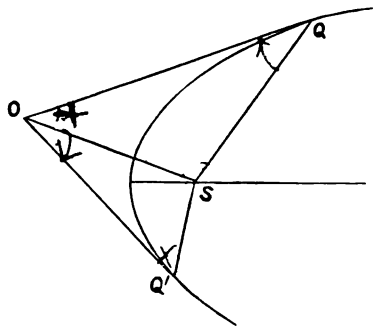
Ex. Draw a diagram, and show that the results proved in this article hold good when O is on the other side of the directrix.

32. Now observe that we have proved *generally* the truth of the three Theorems of which a proof for a *particular* case is given in Art. 28, 29.

For two tangents OQ, OQ' have been drawn to the parabola from the same point O, and we have proved that

I. $\angle OQS = \angle Q'OS$.

That is, the angle between the first tangent and the focal distance of the point Q, *to* which it is drawn, is equal to the angle between the second tangent and the focal distance of the point O, *from* which it is drawn.



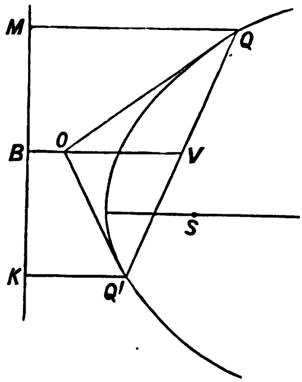
II. $\angle OSQ = \angle OSQ'$.

That is, the tangents subtend equal angles at the focus.

III. $OS^2 = QS \cdot Q'S$.

That is, the focal distance of the point, *from* which the tangents are drawn, is a mean proportional between the focal distances of the points, *to* which the tangents are drawn.

33. *If OQ, OQ' be tangents, and OV be drawn parallel to the axis to meet the chord QQ' in V, then QQ' is bisected in V.*



Let OV meet the directrix in B.

Then, if QM, $Q'K$ be drawn perpendicular to the directrix,

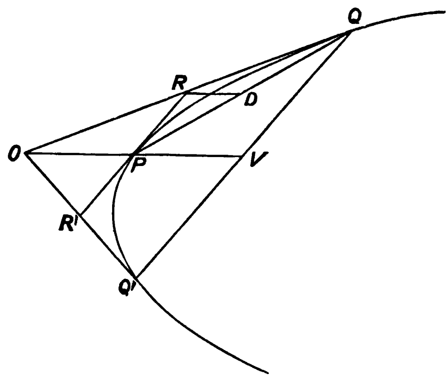
$$MB = KB. \qquad \text{(Art. 31.)}$$

And since MQ, BV, KQ' are parallel,

$$QV : VQ' = MB : BK; \qquad \text{(Art. 8 (1).)}$$

$$\therefore QV = VQ'$$

34. *If OV, drawn as in the preceding proposition, meet the parabola in P, and if RPR' be drawn touching the parabola at P and meeting OQ, OQ' in R, R', then shall RPR' be parallel to QQ', and OV, RR' shall bisect each other in P.*



Join PQ, and draw RD parallel to OV, meeting PQ in D.

Then since RQ, RP are tangents from R,

$\therefore PD = QD$; (Art. 33.)

and $\therefore$ since RD is parallel to OP,

$OR = RQ$. (Eucl. VI. 2.)

Similarly it may be proved that $OR' = R'Q'$,

$\therefore RR'$ is parallel to QQ'. (Eucl. VI. 2.)

Hence $OP = PV$, (Eucl. VI. 2.)

and $QV = 2RP$.

Similarly, $Q'V = 2R'P$,

and $\therefore RP = R'P$.

35. We have seen in Art. 27 that the locus of the middle points of a system of parallel chords is a straight line parallel to the axis of the parabola.

This is also evident from what we have just proved, viz., that the middle point V of a chord QQ', the point P in which a line parallel to the chord touches the curve, and the point O in which the tangents at the ends of the chord meet, are all in one straight line, which is parallel to the axis.

Now for all chords parallel to QQ', P is the same point, and as there can be but one straight line drawn through P parallel to the axis, this must be the line OPV.

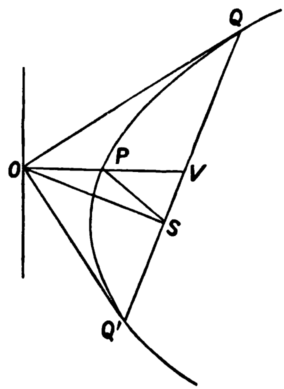
Hence the middle points of all chords parallel to QQ' lie in the line OPV.

DEF. The straight line, which passes through the middle points of a system of parallel chords, is called a *Diameter*.

DEF. The segments of the chords bisected by any diameter are called the *Ordinates* of that diameter.

In the propositions that follow, the word *diameter* is restricted to that portion of a diameter which falls within the parabola: thus we call PV a diameter, P the end of the diameter, and QV, $Q'V$ ordinates of the diameter at the point V.

36. *If the focal chord QSQ' be bisected by the diameter PV, then shall $QQ' = 4SP$.*



Since QSQ' is a focal chord,

(1.) The tangents at Q and Q' meet in O, a point on the directrix, and QOQ' is a right angle; (Art. 24.)

(2.) A line drawn from O parallel to the axis passes through V, the middle point of QQ' (Art. 33), and therefore, since PV is parallel to the axis, OV and PV are in the same straight line;

(3.) Since QOQ' is a right angle, a circle described on QQ' as diameter passes through O;

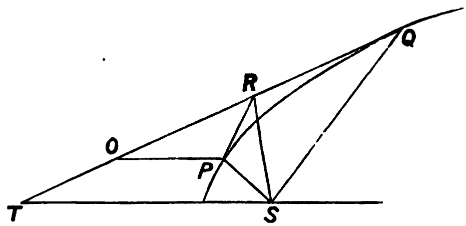
$$\therefore\ QV = Q'V = OV$$

$$\begin{aligned} \text{Then } QQ' &= 2QV \\ &= 2OV \\ &= 4OP \qquad \text{(Art. 34.)} \\ &= 4SP, \end{aligned}$$

a result established, without any reference to the properties of diameters, in Art. 26.

DEF. The focal chord QQ', or its equivalent $4SP$, is called the *Parameter* of the diameter PV, which bisects the chord.

37. *If from a point P on the curve a line PO be drawn parallel to the axis, and the tangent at P meet the tangent from O in R, the triangles PRS, POR are similar.*



Let the tangent QRO meet the axis in T.

Then since RP, RQ are tangents from the same point R.

$$\therefore \angle PRS = \angle RQS \quad \text{(Art. 31.)}$$
$$= \angle QTS \quad \text{(Art. 28.)}$$
$$= \angle POR; \quad \text{(Eucl. I. 29.)}$$

and, since RP produced bisects $\angle OPS$, (Art. 23.)

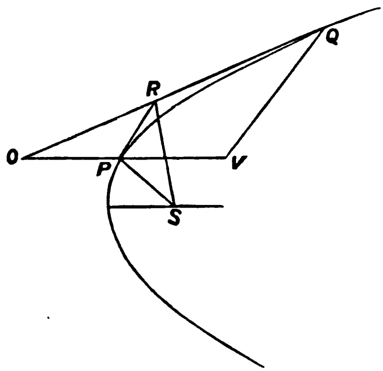
$$\therefore \angle RPS = \angle OPR;$$

$\therefore$ the triangles PRS, POR are similar.

COR. Hence it follows that $SP : RP = RP : PO$,

and $\therefore RP^2 = SP \cdot PO$.

38. *If QV be an ordinate to the diameter PV, then shall* $QV^2 = 4SP \cdot PV$.



Let RP, the tangent at P, meet OQ, the tangent at Q, in R.

Join SR, SP.

Then PRS, POR are similar triangles. (Art. 37.)

$\therefore RP^2 = SP \cdot PO$. (Art. 37.)

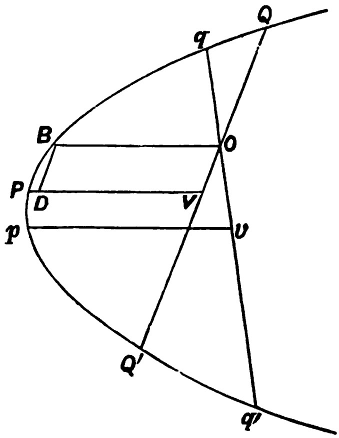
Now $QV = 2RP$; (Art. 34.)

$\therefore QV^2 = 4RP^2$ (Art. 8 (5.))

$= 4SP \cdot PO$

$= 4SP \cdot PV$. (Art. 34.)

39. *If two chords of a parabola intersect each other, the rectangles contained by their segments are in the ratio of the parameters of the diameters bisecting the chords.*



Let the chords QQ', qq' intersect in O.

Draw the diameters PV, pv, bisecting the chords in V, v.

Draw OB parallel to PV, and BD parallel to QQ'.

Then $QO \,.\, Q'O = QV^2 - OV^2$ (Eucl. II. 5)

$= QV^2 - BD^2$ (Eucl. I. 34.)

$= 4SP \,.\, PV - 4SP \,.\, PD$ (Art. 38.)

$= 4SP\,(PV - PD)$

$= 4SP \,.\, BO.$

Similarly, $qO \,.\, q'O = 4Sp \,.\, BO$;

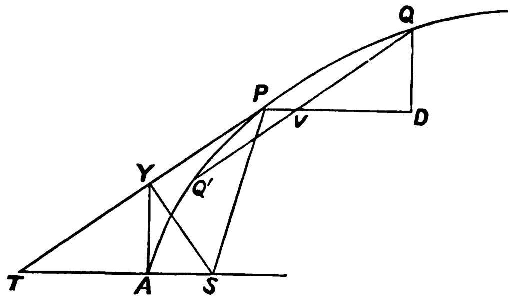
$\therefore\ QO \,.\, Q'O : qO \,.\, q'O = 4SP \,.\, BO : 4Sp \,.\, BO$

$= 4SP : 4Sp$

$=$ parameter of PV : parameter of pv.

Ex. Show that this proposition is true when O lies outside the parabola.

40. *If from either end of a chord QQ', a perpendicular QD be drawn to the diameter PV bisecting the chord, then* $QD^2 = 4AS.PV$.



Let AY be the tangent at the vertex, meeting PT, the tangent at P, in Y.

Then QVD and YTA are similar triangles, (Art. 8 (6).)

and YTA and SYA are similar triangles; (Eucl. VI. 8.)

$\therefore$ QVD, SYA are similar triangles; (Eucl. VI. 21.)

and $\therefore$ $QD^2 : QV^2 = AS^2 : SY^2$

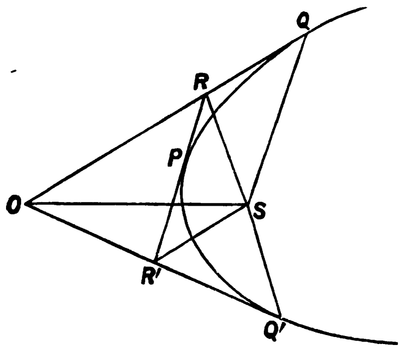
$= AS^2 : AS.SP$ (Art. 28.)

$= AS : SP$.

Hence $QD^2 : 4SP.PV = AS : SP$, (Art. 38.)

and $\therefore$ $QD^2 = 4AS.PV$.

41. *The circle passing through the points of intersection of three tangents to a parabola also passes through the focus.*



Let OQ, OQ', RR' be the tangents,

Q, Q', P the points of contact,

O, R, R' the points of intersection,

S the focus.

Then $\angle R'RS = \angle OQS$ (Art. 31.)

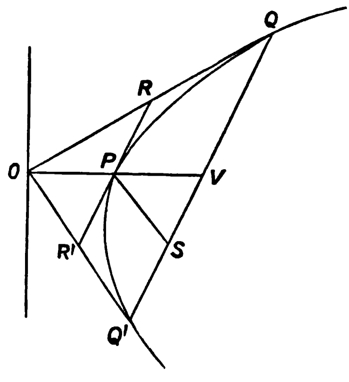
$= \angle R'OS.$ (Art. 31.)

Hence a circle can be described passing through O, R, S, R' (Eucl. III. 21); that is, the focus S lies on the circle described about the triangle ROR'.

42. *The Five Point Circle.*

Let QQ' be a focal chord; OQ, OQ' tangents at Q, Q', meeting at right angles in O, a point on the directrix; PV a diameter passing through O; RPR' the tangent at P, meeting the other tangents in R, R'.

Then shall O, R, V, S, R' lie on the circumference of a circle of which the centre is P.



For RR' is bisected in P; (Art. 34.)

and since ROR' is a right angle,

$\therefore$ P is the centre of the circle described about ORR';

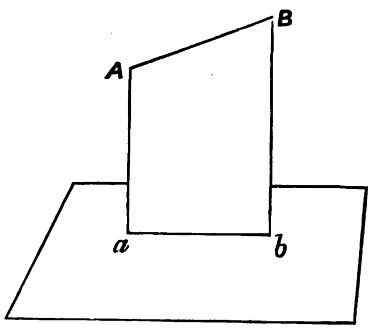
and since $SP = OP = VP$, (Art. 34.)

$\therefore$ this circle passes through V and S.

CHAPTER III.

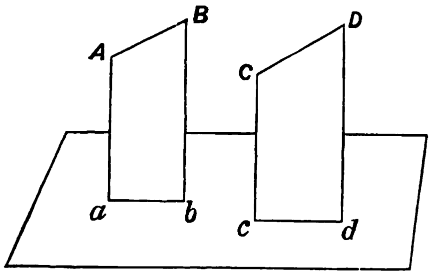
Orthogonal Projection.

43. DEF. If from a point A, a perpendicular Aa be drawn to a plane, the point a is called the *Projection* of A on the plane.



If AB be a straight line, and a, b the projections of A, B on a plane, the straight line ab is the projection of AB on the plane. For ab is the common section of the plane of projection with a plane perpendicular to it and passing through AB, and therefore the projection of every point in AB lies in ab.

44. *If AB, CD be two parallel straight lines, and ab, cd their projections on the same plane, then will ab and cd be parallel.*



For the plane passing through BAa is parallel to the plane passing through DCc, because AB is parallel to CD, and Aa is parallel to Cc, by Eucl. XI. 15.

And since these parallel planes are cut by the plane of projection, their common sections with it are parallel, by Eucl. XI. 16;

$\therefore$ ab is parallel to cd.

45. *If ab, cd be the projections on the same plane of parallel straight lines AB, CD, then $ab : cd = AB : CD$.*

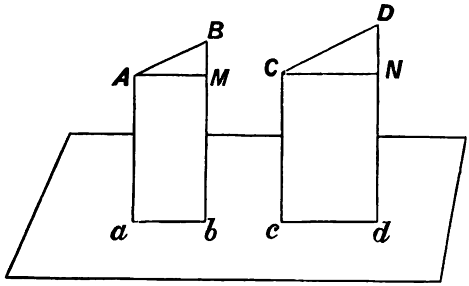
Draw AM parallel to ab, and CN parallel to cd.

Then $AM = ab$, an d$CN = cd$.

Also AM being parallel to ab, which is parallel to cd, which is parallel to CN,

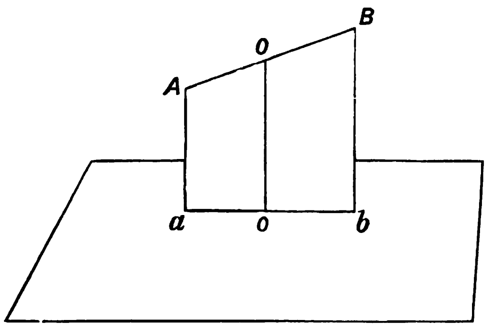
$\therefore$ AM is parallel to CN. (Eucl. XI. 9.)

Hence, since AB is also parallel to CD,
and BM is also parallel to DN,
the triangles ABM, CDN are equiangular; (Eucl. XI. 10.)



$$\therefore AB : CD = AM : CN = ab : cd.$$

46. *The segments of a divided line are to one another in the same ratio as the segments of its projection.*



Let aob be the projection of the straight line AOB, a, o, b being the projections of the points A, O, B.

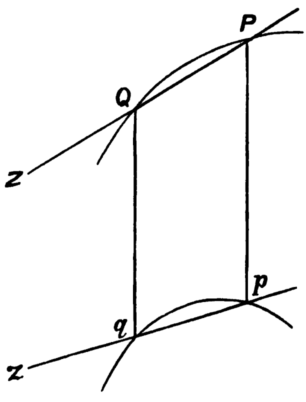
Then, since Aa, Oo, Bb are parallel,

$$AO : OB = ao : ob. \qquad \text{(Art. 8 (1).)}$$

Projection of a Curve.

47. Def. If from all the points of a curve perpendiculars be drawn to a plane, the curve formed by the feet of the perpendiculars is the projection of the given curve on that plane.

48. If P and Q be points near to each other on a curve, and p, q be the projections of P, Q, then as Q is made to approach P, the perpendicular Qq approaches Pp, and ultimately when Q coincides with P, Qq will coincide with Pp.

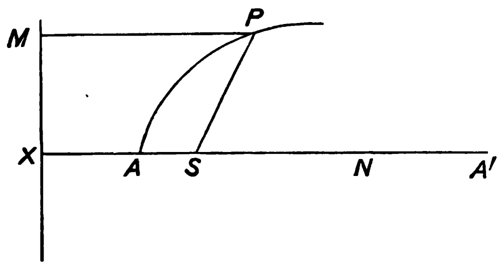


Thus, when the secant ZQP becomes the tangent at P, the secant zqp becomes the tangent at p, that is, a tangent projects into a tangent.

CHAPTER IV.

The Ellipse.

49. Def. If a point P move in such a way that its distance SP from a fixed point S is in a constant ratio, less than unity, to its perpendicular distance PM from a fixed straight line MX, the curve traced out by P is called an *Ellipse.*



The point S is called a *Focus;* the line MX is called a *Directrix.*

Through S draw XSN perpendicular to the directrix. The line XSN is called the *Axis.*

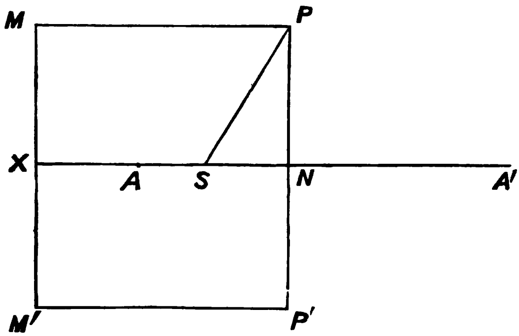
The constant ratio of $SP : PM$ is called the *Eccentricity.*

In SX take a point A, such that $SA : AX = SP : PM$; then A is a point in the ellipse. This point is called a *Vertex.*

In XS produced, take a point A', such that $SA' : A'X = SP : PM$; then A' is a point in the ellipse. This point is likewise called a *Vertex.*

50. *To construct the curve by the determination of successive points in it.*

Through N, any point in AA', draw a straight line perpendicular to AA', and take SP of such a length that $SP : NX = SA : AX$. (Eucl. VI. 12.)



With S as centre, and radius SP, describe a circle cutting the straight line in P, P'. Draw PM, $P'M'$, perpendiculars to the directrix.

$$\text{Then } SP : PM = SP : NX = SA : AX;$$

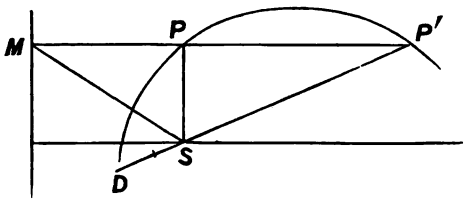
$$\therefore P \text{ is a point on the curve.}$$

Similarly P' is a point on the curve.

In this way, by taking N at other points in the line AA', we might determine any number of points in the curve.

And since $PN = P'N$, the curve will be symmetrical with regard to the axis AA'.

51. *For any point P on the curve there is a corresponding point P′ on the curve, in the line drawn through P perpendicular to the directrix, and on the same side of the directrix as P.*



Join SM, and make the angle MSD equal to the angle MSP, and let DS produced meet MP or MP produced in P'.

Then MS bisects the exterior angle of the triangle PSP',

$$\text{and } \therefore SP' : SP = P'M : PM; \qquad \text{(Eucl. VI. A.)}$$

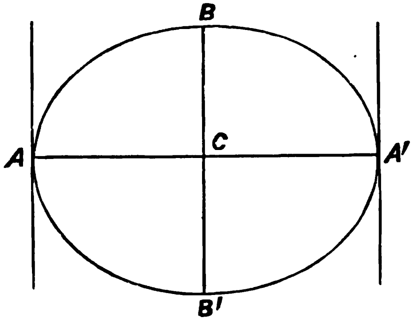
$$\therefore SP' : P'M = SP : PM$$

$$= \text{the eccentricity};$$

$\therefore P'$ is a point on the ellipse.

52. *Description of the form of the curve.*

The *Ellipse* is a closed curve lying between two lines drawn through A and A' perpendicular to the axis.

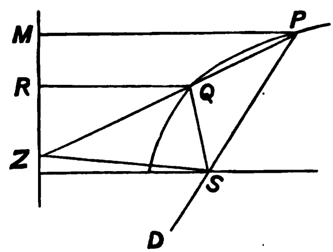


Bisect AA' in C. C is called the *Centre* of the ellipse.

Draw BCB', meeting the curve in B, B', perpendicular to AA'.

Then AA' is called the *Major Axis*;
and BB' is called the *Minor Axis*.

53. *If the chord PQ meet the directrix in Z, then ZS will bisect QSD, the exterior angle between PS and QS.*



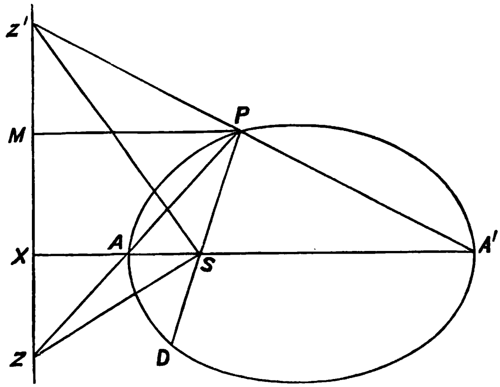
Draw PM and QR perpendiculars to the directrix.

Then $PS : QS = PM : QR$, from the definition of the curve,

$= PZ : QZ$; (Eucl. VI. 2.)

and $\therefore ZS$ bisects the angle QSD. (Eucl. VI. A.)

54. *If two chords, passing through the vertices and any point in the curve, meet the directrix in Z, Z', then ZSZ' is a right angle.*



Let two chords PA, $A'P$, passing through the vertices and any point P in the curve, meet the directrix in Z, Z'.

Produce PS to D, and draw PM perpendicular to the directrix.

Then $PS : AS = PM : AX$, from the definition of the curve,

$= PZ : AZ$; (Eucl. VI. 2.)

and $\therefore ZS$ bisects $\angle ASD$. (Eucl. VI. A.)

And $PS : A'S = PM : A'X$, from the definition of the curve,

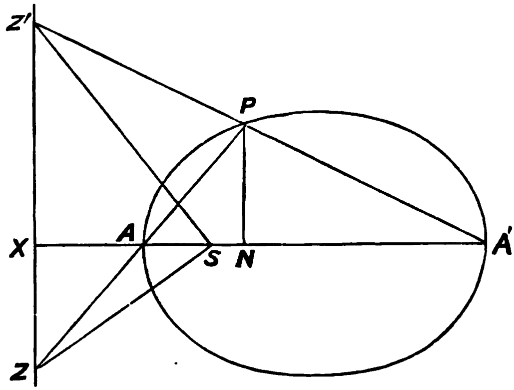
$= PZ' : A'Z'$; (Eucl. VI. 2.)

and $\therefore Z'S$ bisects $\angle PSA$; (Eucl. VI. A.)

and $\therefore ZSZ'$ is a right angle, (Art. 8 (4).)

COR. Hence $ZX \cdot Z'X = SX^2$. (Eucl. VI. 8.)

55. *If PN be the ordinate of a point P on the curve, then* $PN^2 : AN . A'N = CB^2 : CA^2$.



Let the chords PA, $A'P$ meet the directrix in Z, Z'.

Then, by similar triangles PAN, ZAX,

$$PN : AN = ZX : AX; \qquad (1.)$$

and by similar triangles $PA'N$, $Z'A'X$,

$$PN : A'N = Z'X : A'X. \qquad (2.)$$

Then, compounding the proportions (1) and (2),

$$PN^2 : AN . A'N = ZX . Z'X : AX . A'X$$
$$= SX^2 : AX . A'X \qquad \text{(Art. 54.)}$$

= a constant ratio for all positions of P.

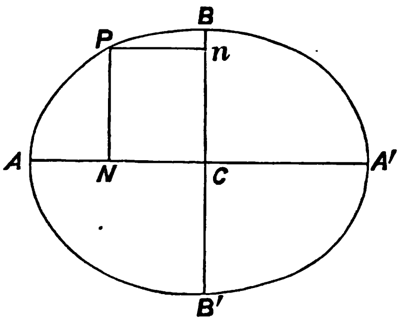
Now suppose B, the end of the minor axis, to be the point from which the ordinate is drawn. Then (as in the diagram of Art. 56) BC is the ordinate, and

$$CB^2 : AC.A'C = \text{the constant ratio};$$

that is, $CB^2 : CA^2 =$ the constant ratio.

Hence $PN^2 : AN . A'N = CB^2 : CA^2$.

56. *If Pn be drawn perpendicular to the minor axis, then* $Pn^2 : Bn \,.\, B'n = CA^2 : CB^2$.



We have $Bn . B'n = CB^2 - Cn^2$ (Eucl. II. 5.)

$= CB^2 - PN^2$;

and $AN . A'N = CA^2 - CN^2$ (Eucl. II. 5.)

$= CA^2 - Pn^2$.

Now $PN^2 : AN \,.\, A'N = CB^2 : CA^2$; (Art. 55.)

$\therefore PN^2 : CA^2 - Pn^2 = CB^2 : CA^2$;

$\therefore PN^2 : CB^2 = CA^2 - Pn^2 : CA^2$;

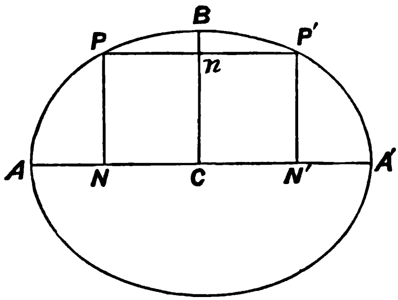
$\therefore CA^2 : CA^2 - Pn^2 = CB^2 : PN^2$;

$\therefore CA^2 : Pn^2 = CB^2 : CB^2 - PN^2$; (Eucl. v. E.)

$\therefore CA^2 : CB^2 = Pn^2 : Bn \,.\, B'n$;

that is, $Pn^2 : Bn \,.\, B'n = CA^2 : CB^2$.

57. *The Ellipse is symmetrical with respect to the minor axis.*



Take $A'N' = AN$; draw the ordinates PN, $P'N'$; and join PP', cutting CB in n.

Then $PN^2 : AN \,.\, A'N = CB^2 : CA^2$; (Art. 55.)

and $P'N'^2 : AN' \,.\, A'N' = CB^2 : CA^2$; (Art. 55.)

and therefore, since $A'N' = AN$, and $A'N = AN'$,

$$PN^2 = P'N'^2;$$

and $\therefore PN = P'N'$.

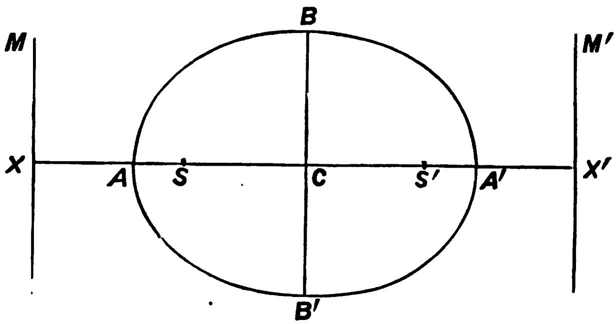
Hence PP' is parallel and equal to NN'; (Eucl. I. 33.) and $\therefore CB$ is perpendicular to PP'.

Then, since NN' is bisected in C,

PP' is bisected in n. (Art. 8 (1).)

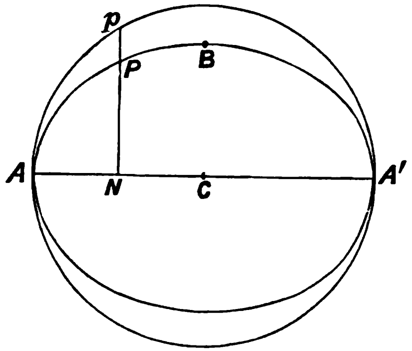
Thus the curve is symmetrical with respect to the minor axis.

58. *The Ellipse has a second focus and a second directrix.*



Since the ellipse is symmetrical with respect to each of the axes, AA', BB', if in CA' we take $CS' = CS$, and in CA' produced we take $CX' = CX$, and draw $M'X'$ perpendicular to CX', the curve may be described by means of the focus S' and the directrix $M'X'$ exactly in the same way as by means of the focus S and the directrix MX.

59. *If the ordinate PN meets the circle described on the major axis as diameter in p, then $PN : pN = CB : CA$.*



With centre C and diameter AA' describe a circle. This is called the *Auxiliary Circle.*

Let the ordinate PN produced meet the circle in p.

Then $pN^2 = AN \,.\, A'N.$ (Eucl. III. 35.)

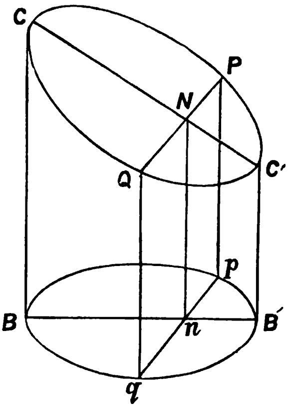
Now $PN^2 : AN \,.\, A'N = CB^2 : CA^2$; (Art. 55.)

$\therefore PN^2 : pN^2 = CB^2 : CA^2$;

$\therefore PN : pN = CB : CA.$

Conversely, if ApA' be any circle of which AA' is a diameter, and C the centre, and if the ordinates of points on the circumference, as pN, be divided in the ratio of CB to CA, CB being less than CA, the locus of the points of division will be an ellipse, whose major axis is AA', and whose minor axis is equal to $2CB$.

60. *The Projection of a Circle is an Ellipse.*



Let $BpB'q$ be the projection of the circle $CPC'Q$.

Let CC' be a diameter of the circle at right angles to the straight line in which the plane of the circle meets the plane of projection; and let BB' be the projection of CC'.

Let a plane $PQqp$, perpendicular to the plane of projection, and placed so that PQ is parallel to the straight line in which the plane of the circle meets the plane of projection, cut CC' in N and BB' in n.

Then $PQ=pq$, and CC', BB' bisect PQ, pq in N and n at right angles.

Then, since CB, Nn, $C'B'$ are parallel,

$$CN : Bn = CC' : BB'; \qquad \text{(Art. 8 (1).)}$$

$$\text{and } C'N : B'n = CC' : BB'; \qquad \text{(Art. 8 (1).)}$$

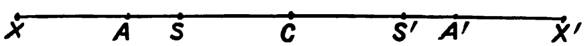
$$\therefore CN \,.\, C'N : Bn \,.\, B'n = CC'^2 : BB'^2.$$

$$\text{But } CN \,.\, C'N = PN^2 = pn^2; \qquad \text{(Eucl. III. 35.)}$$

$$\therefore pn^2 : Bn \,.\, B'n = CC'^2 : BB'^2.$$

Hence (by Art. 56) $BpB'q$ is an ellipse having its axes in the ratio $CC' : BB'$.

61. *To show that* (1) $CS : CA =$ *the eccentricity;*
(2) $CA : CX =$ *the eccentricity;*
(3) $CS \,.\, CX = CA^2$.



(1.) Since $SA' : A'X = SA : AX$,

$\therefore\ SA' : SA = A'X : AX$;

$\therefore\ SA' - SA : SA = A'X - AX : AX$;

$\therefore\ SA' - S'A' : SA = AA' : AX$;

$\therefore\ SS' : SA = AA' : AX$;

$\therefore\ SS' : AA' = SA : AX$;

$\therefore\ CS : CA = SA : AX$;

that is, $CS : CA =$ the eccentricity.

(2.) Since $SA' : A'X = SA : AX$,

$\therefore\ SA' : SA = A'X : AX$;

$\therefore\ SA' + SA : SA = A'X + AX : AX$;

$\therefore\ AA' : SA = A'X + A'X' : AX$;

$\therefore\ AA' : SA = XX' : AX$;

$\therefore\ AA' : XX' = SA : AX$;

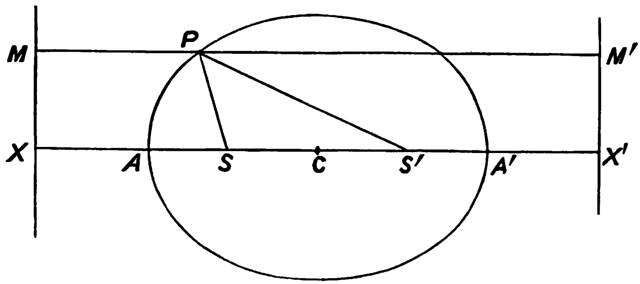
$\therefore\ CA : CX = SA : AX$;

that is, $CA : CX =$ the eccentricity.

(3.) Since $CS : CA$ and $CA : CX$ are ratios each of which is equal to the eccentricity,

$\therefore\ CS : CA = CA : CX$; and $\therefore\ CS \,.\, CX = CA^2$.

62. *The sum of the focal distances of any point on the ellipse is equal to the major axis.*



We have to show that $SP+S'P=AA'$.

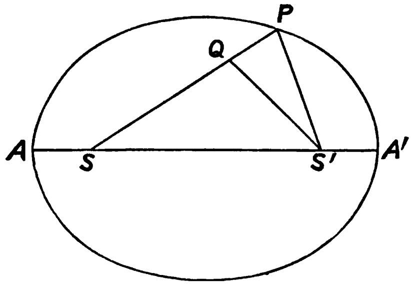
Draw MPM' to meet the directrices at right angles.

$$\begin{aligned}
\text{Then } SP : PM &= S'P : PM'; \\
\therefore\ SP : S'P &= PM : PM'; \\
\therefore\ SP+S'P : SP &= PM+PM' : PM; \\
\therefore\ SP+S'P : PM+PM' &= SP : PM; \\
\therefore\ SP+S'P : MM' &= SP : PM \\
&= CA : CX \qquad \text{(Art. 61.)} \\
&= 2CA : 2CX \\
&= AA' : XX'.
\end{aligned}$$

$$\text{But } MM' = XX';$$

$$\therefore\ SP+S'P=AA'.$$

63. *The sum of the focal distances of a point within the ellipse is less than the major axis.*



Let Q be any point within the ellipse.

Join SQ, $S'Q$; produce SQ to meet curve in P. Join $S'P$.

Then $S'P+PQ$ is greater than $S'Q$; (Eucl. I. 20.)

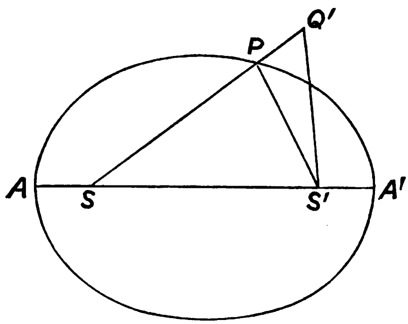
$\therefore$ $S'P+PQ+SQ$ is greater than $S'Q+SQ$;

$\therefore$ $S'P+SP$ is greater than $S'Q+SQ$.

But $S'P+SP=AA'$; (Art. 62.)

$\therefore$ AA' is greater than $S'Q+SQ$.

64. *The sum of the focal distances of a point without the ellipse is greater than the major axis.*



Let Q' be any point without the ellipse.

Join SQ', $S'Q'$, and let P be the point where SQ' cuts the curve.

Join $S'P$.

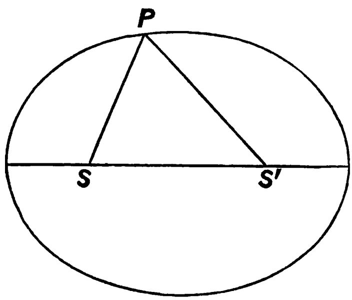
Then $S'Q'+Q'P$ is greater than $S'P$; (Eucl. I. 20.)

$\therefore$ $S'Q'+Q'P+PS$ is greater than $S'P+SP$;

$\therefore$ $S'Q'+SQ'$ is greater than $S'P+SP$;

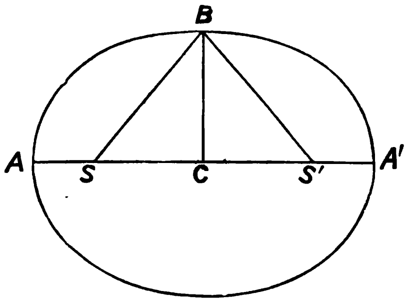
$\therefore$ $S'Q'+SQ'$ is greater than AA'.

65. *The mechanical construction of an Ellipse.*



If we fasten the two ends of a string at two points S, and S', the string being longer than SS', and trace a curve on the paper, with a pencil pressed against the string to keep it tight, the curve traced will be an ellipse whose major axis is equal in length to the string.

66. *To prove that $CB^2 = AS \cdot A'S$.*



If BC be the semi-axis minor, it is plain that $SB = S'B$.

$$\text{Now } SB + S'B = AA'; \qquad \text{(Art. 62).}$$

$$\therefore 2SB = 2CA;$$

$$\therefore SB = CA.$$

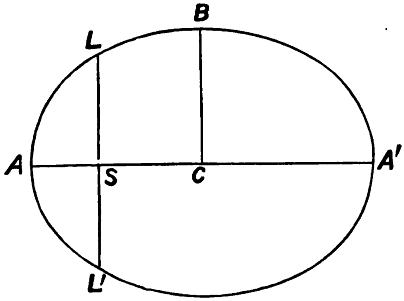
$$\text{Hence } CB^2 = SB^2 - CS^2 \qquad \text{(Eucl. I. 47.)}$$
$$= CA^2 - CS^2.$$

Now since AA' is divided equally in C and unequally in S,

$$AS \cdot A'S = CA^2 - CS^2. \qquad \text{(Eucl. II. 5.)}$$

$$\therefore CB^2 = AS \cdot A'S.$$

67. *To find an expression for the Latus Rectum.*



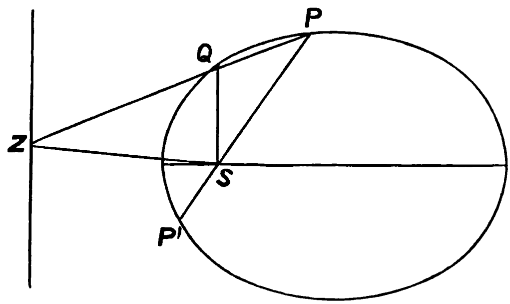
DEF. The double ordinate LL' through either focus is called the *Latus Rectum.*

$$\text{Now } LS^2 : AS \,.\, A'S = CB^2 : CA^2; \qquad \text{(Art. 55.)}$$
$$\therefore LS^2 : CB^2 = CB^2 : CA^2; \qquad \text{(Art. 66.)}$$
$$\therefore LS : CB = CB : CA;$$
$$\therefore LS \,.\, CA = CB^2;$$
$$\text{or, } LS = \frac{CB^2}{CA}.$$
$$\text{Hence the } \textit{Latus Rectum} = \frac{2CB^2}{CA}.$$

The Tangent.

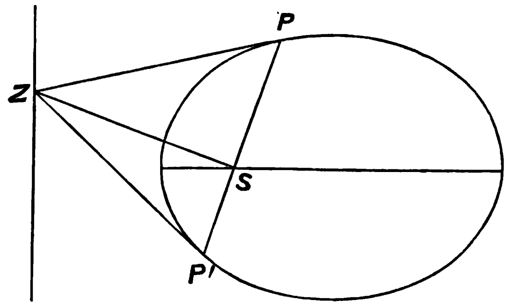
68. A Secant to the ellipse cuts the curve in two points only. The proof of this is precisely the same as that given in Art. 21 for the same property in the parabola; and, as in the parabola, a tangent to an ellipse is the ultimate position of a secant when the two points in which the secant cuts the curve have become coincident. (See Art. 22.)

69. *If the tangent at P meets the directrix in Z, then ZSP is a right angle; and the tangents at the extremities of the focal chord PSP' intersect on the directrix.*



Let the chord PQ meet the directrix in Z. Produce PS to meet the curve in P'; then ZS bisects $\angle QSP'$. (Art. 53.)

Now let Q move up to P; then, when Q coincides with P, the angle QSP' becomes equal to two right angles, and ZS is perpendicular to PP'.

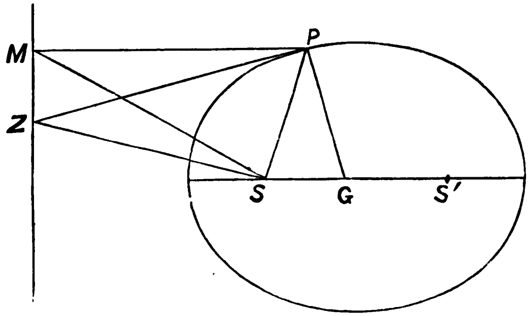


In this position ZP is the tangent at P.

Also ZP' will be the tangent at P', and thus the tangents at P, P' intersect on the directrix.

Note.—In the same way it may be shown that, if the tangent at P meets the other directrix in Z', the angle $Z'S'P$ is a right angle.

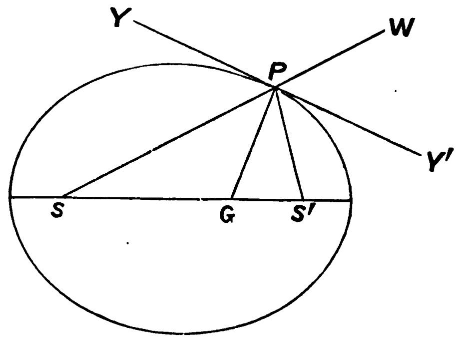
70. *If the normal at P meets the major axis in G, then $SG : SP =$ the eccentricity.*



Let PZ be the tangent at P, meeting the directrix in Z.
Draw PM perpendicular to the directrix.
Since PMZ and PSZ are right angles, a circle described on PZ as diameter will pass through M and S,
and PG, which is perpendicular to PZ,
will touch the circle at P. (Eucl. III. 16.)

Hence $\angle SPG = \angle PMS$ in alternate segment (Eucl. III. 32)
and $\angle PSG = \angle MPS$, since PM is parallel to SG;
$\therefore$ SPG, SMP are similar triangles,
and $\therefore$ $SG : SP = SP : PM$;
that is, $SG : SP =$ the eccentricity.
Similarly, by producing ZP and MP to meet the other directrix, it may be shown that
$S'G : S'P =$ the eccentricity.
Hence $SG : S'G = SP : S'P$.

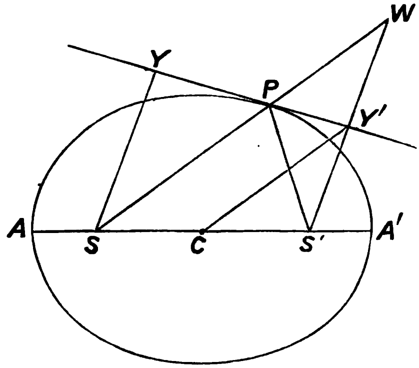
71. *The tangent and normal at P respectively bisect the exterior and interior angles between the focal distances of P.*



Produce SP to W.
Let YPY' be the tangent at P,
and PG the normal at P.
Then $SG : S'G = SP : S'P$; (Art. 70.)
and $\therefore$ GP bisects the angle SPS'; (Eucl. VI. 3.)
that is, the normal bisects the angle between
the focal distances of P.

And PY' is perpendicular to PG;
$\therefore$ PY' bisects the angle WPS'; (Art. 8 (4).)
that is, the tangent bisects the exterior angle
between the focal distances of P.
Also, since $\angle\, YPS = \angle\, WPY'$, (Eucl. I. 15.)
$\therefore$ $\angle\, YPS = \angle\, Y'PS'$;
that is, the tangent makes equal angles with
the focal distances of P.

72. *The perpendiculars from the foci on the tangent intersect the tangent in the circumference of the auxiliary circle.*



Let SY, $S'Y'$ be perpendiculars on the tangent at P.

Produce SP to meet $S'Y'$ produced in W.

Then since $\angle S'PY' = \angle WPY'$, (Art. 71.)

and $\angle PY'W = \angle S'Y'P$, being right angles,

$\therefore PW = S'P$; (Eucl. I. 26.)

$$\text{and } \therefore SW = SP + PW$$
$$= SP + S'P$$
$$= 2CA.$$

Now join CY', which bisects SS' and WS';

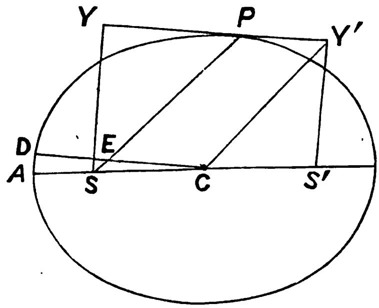
$\therefore CY'$ is parallel to SW, (Eucl. VI. 2.)

$$\text{and } \therefore CY' = \frac{1}{2} . SW = CA.$$

Hence Y' is a point on the auxiliary circle.

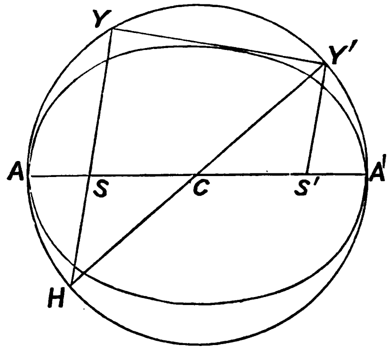
Similarly Y may be proved to be a point on that circle, and CY may be proved to be parallel to $S'P$.

73. *If CE, parallel to the tangent at P, meets SP in E, then $PE = CA$.*



Draw CD parallel to the tangent YPY',
meeting the curve in D, and cutting SP in E.
Then $PECY'$ is a parallelogram, (Art. 72.)
and $\therefore PE = CY' = CA$.

74. *To show that $SY.S'Y' = CB^2$.*



Produce YS to meet the auxiliary circle in H;
Then since HYY' is a right angle,
HY' is a diameter passing through C,
and $\therefore CH = CY'$.

But $SC = S'C$, and $\angle SCH = \angle Y'CS'$;

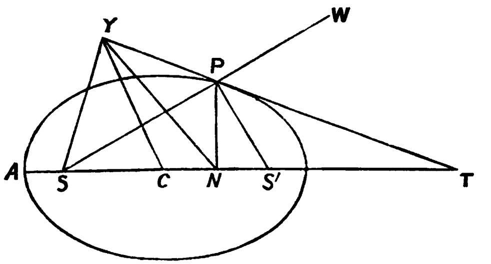
$\therefore SH = S'Y'$. (Eucl. I. 4.)

And $SY . S'Y' = SY . SH$

$= AS . A'S$ (Eucl. III. 35.)

$= CB^2$. (Art. 66.

75. *If the tangent at P meets the major axis produced in T, then* $CN . CT = CA^2$.



Draw SY perpendicular to PT.

Then, since PNS and PYS are right angles,

A circle can be described round $PNSY$;

and $\therefore \angle SNY = \angle SPY$.

Then we can show that CNY, CYT are similar triangles;

for they have a common angle at C,

and $\angle CNY = \angle SNY$

$= \angle SPY$

$= \angle S'PT$ (Art. 71.)

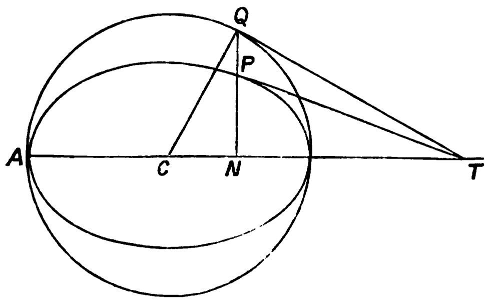
$= \angle CYT$; (Art. 72.)

$\therefore CN : CY = CY : CT$;

$\therefore CN . CT = CY^2$

$= CA^2$. (Art. 72.)

76. *If QPN be an ordinate of the auxiliary circle, and PN an ordinate of the ellipse, the tangents at Q and P meet the major axis in the same point.*



Let PT be the tangent to the ellipse at P.

Join CQ, QT.

Then $CN \, . \, CT = CA^2$ (Art. 75.)

$= CQ^2$,

and $\therefore CN : CQ = CQ : CT$.

Hence the triangles CNQ, CQT are similar (Eucl. VI. 6.),

and $\therefore CQT$ is a right angle,

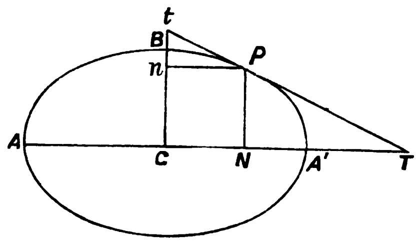
and $\therefore QT$ touches the circle at Q. (Eucl. III. 16.)

Thus the tangents at P and Q cut the major axis produced in the same point.

Observe also that

$CN \, . \, NT = QN^2$. (Eucl. VI. 8.)

77. *If the tangent at P meets the minor axis produced in t, and Pn be drawn perpendicular to the minor axis, then* $Cn \,.\, Ct = CB^2$.



By similar triangles tCT, PNT,

$$Ct : CT = PN : NT;$$

and $Cn : CN = PN : CN$,

because $PnCN$ is a rectangle;

$$\therefore Cn \,.\, Ct : CN \,.\, CT = PN^2 : CN \,.\, NT;$$

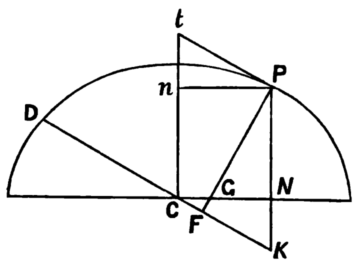
$$\therefore Cn \,.\, Ct : CA^2 = PN^2 : CN \,.\, NT$$

$$= PN^2 : QN^2 \qquad \text{(Art. 76.)}$$

$$= CB^2 : CA^2; \qquad \text{(Art. 59.)}$$

$$\therefore Cn \,.\, Ct = CB^2.$$

78. *If PG, the normal at P, meets a line, drawn through C parallel to the tangent at P, in F, then $PF \,.\, PG = CB^2$.*



Let the ordinate PN meet CD, drawn through C parallel to Pt, in K.

Then, since the angles at N and F are right angles,

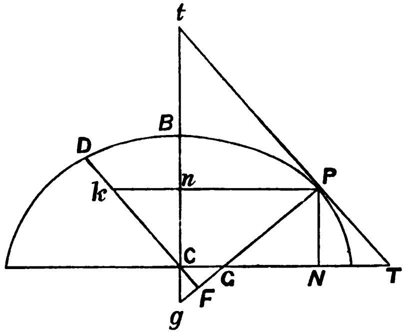
a circle can be described round $FGNK$;

$$\therefore PF \,.\, PG = PN \,.\, PK \qquad \text{(Eucl. III. 36.)}$$

$$= Cn \,.\, Ct$$

$$= CB^2. \qquad \text{(Art. 77.)}$$

79. *If PG meets the minor axis in g, then $PF.Pg = CA^2$.*



Since the angles at n and F are right angles,
a circle can be described round $Fgkn$;

$$\therefore PF.Pg = Pn.Pk \qquad \text{(Eucl. III. 36.)}$$
$$= CN.CT$$
$$= CA^2. \qquad \text{(Art. 75.)}$$

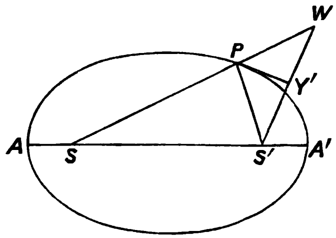
80. *To show that $GN : CN = CB^2 : CA^2$.*

Since $PF.PG = CB^2$,
and $PF.Pg = CA^2$,
$\therefore PG : Pg = CB^2 : CA^2$;
$\therefore GN : CN = CB^2 : CA^2$.

81. *To show that $CG : CN = CS^2 : CA^2$.*

Since $CN : GN = CA^2 : CB^2$, (Art. 80.)
$\therefore CN - GN : CN = CA^2 - CB^2 : CA^2$;
$\therefore CG : CN = CS^2 : CA^2$.

82. *To draw a tangent to the ellipse from a given point on the curve.*



Let P be the given point.

Join SP, $S'P$, and produce SP to W, making $PW = PS'$

Join WS', and draw PY' bisecting the angle WPS'.

Then PY' is the tangent at P. (Art. 71.)

Ex. 1. Show that the circles described on SP and $S'P$ as diameters touch the auxiliary circle.

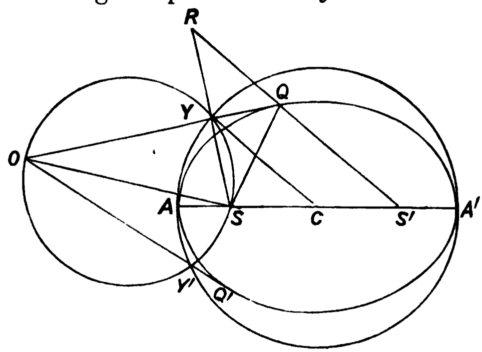
Ex. 2. Show that if O be a point between the ellipse and the auxiliary circle, the circle described on SO as diameter will cut the auxiliary circle.

83. *To draw a pair of tangents to the ellipse from an external point.*

Let O be the given point. Join SO, and on it, as diameter, describe a circle cutting the auxiliary circle in Y, Y'.

Join OY, OY'; these produced will touch the ellipse.

Produce SY to R, making $YR = SY$, and join $S'R$, cutting OY produced in Q. Join CY.



Then, since CY bisects SS' and SR,

$\therefore$ CY is parallel to $S'R$; (Eucl. VI. 2.)

$\therefore$ $S'R = 2CY$.

Again, since $\angle\, OYS$, being the angle in a semi-circle, is a right angle,

$\therefore$ $\angle\, QYR = \angle\, QYS$;

and since $RY = SY$,

$\therefore$ $SQ = QR$; (Eucl. I. 4.)

$$\begin{aligned} \text{and } \therefore\ SQ + S'Q &= QR + S'Q \\ &= S'R \\ &= 2CY \\ &= AA'; \end{aligned}$$

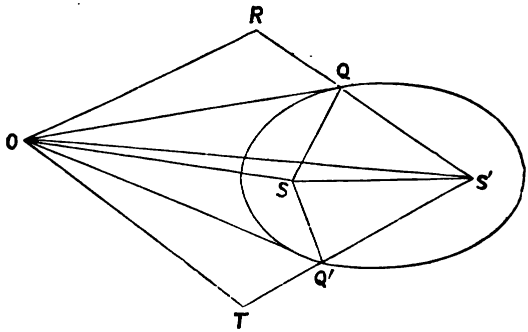
and $\therefore$ Q is a point on the ellipse.

And $\therefore$ since SY, perpendicular to OQ, meets OQ on the auxiliary circle,

OQ is a tangent to the ellipse at Q. (Art. 72.)

Similarly we may show that OY' produced will be a tangent to the ellipse at Q'.

84. *If OQ, OQ' be tangents from O, then $\angle OSQ = \angle OSQ'$.*



Produce $S'Q$ to R, making $QR = QS$, and
produce $S'Q'$ to T, making $Q'T = Q'S$.
Then $S'R = S'Q + QS = AA' = SQ' + Q'S' = S'T$.
And since OQ bisects $\angle RQS$, and $QR = QS$,
$\therefore$ the triangles ORQ, OSQ are equal in all respects;
$\therefore OR = OS$; and similarly $OT = OS$;
and $\therefore OR = OT$.
Hence the sides of ORS', OTS' are equal each to each;

$$\therefore \angle ORS' = \angle OTS',$$

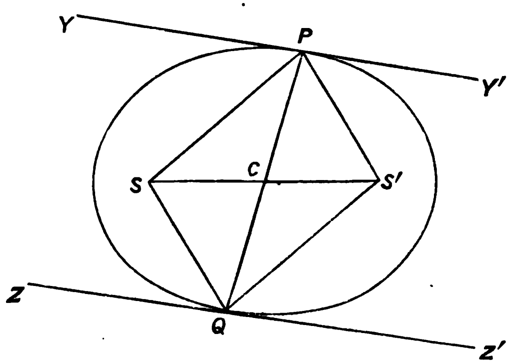
that is, $\angle ORQ = \angle OTQ'$;
and $\therefore \angle OSQ = \angle OSQ'$.

85. *If OQ, OQ' be tangents from O, then $\angle QOS = \angle Q'OS'$.*

For, the same construction being made,

$$\begin{aligned}\angle QOS &= \tfrac{1}{2}\angle ROS \\ &= \tfrac{1}{2}(\angle ROS' + \angle SOS') \\ &= \tfrac{1}{2}(\angle TOS' + \angle SOS') \\ &= \tfrac{1}{2}(\angle TOS + 2\angle SOS') \\ &= \angle Q'OS + \angle SOS' \\ &= \angle Q'OS'.\end{aligned}$$

86. *The centre bisects all chords that pass through it; and the tangents at the ends of a chord passing through the centre are parallel.*



Let P be a point in the ellipse. Join SP, $S'P$.

Complete the parallelogram $SPS'Q$.

Then, since $SQ+S'Q=SP+S'P$, (Eucl. I. 34.)

$\therefore$ Q is a point in the ellipse.

Now the diagonals of a parallelogram bisect each other.

But C, the middle point of SS', is the centre of the ellipse.

$\therefore$ PQ is bisected in the centre of the ellipse.

Next, let YPY', ZQZ' be tangents at P and Q.

Then, since $\angle SPS'=\angle SQS'$, (Eucl. I. 34.)

$\therefore$ the sum of the equal angles YPS, $Y'PS'$ (Art. 71.)

$=$ the sum of the equal angles ZQS, $Z'QS'$; (Art. 71.)

$\therefore$ $\angle YPS=\angle Z'QS'$;

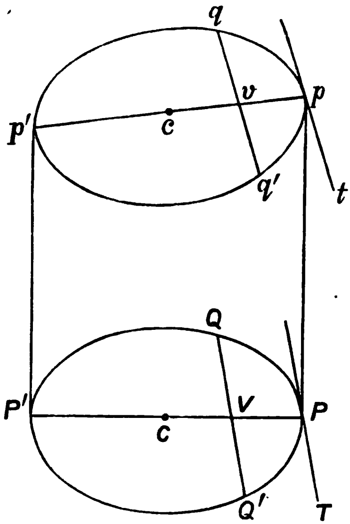
and, adding to each the equal angles SPQ, $S'QP$,

$\therefore$ $\angle YPQ=\angle Z'QP$;

$\therefore$ YY' is parallel to ZZ'.

Diameters.

87. *The locus of the middle points of any system of parallel chords is a straight line passing through the centre of the ellipse.*



Let the ellipse PQP' be the projection of the circle pqp'.

Let qvq' be one of a system of chords in the circle parallel to the tangent at p; let pcp' be a diameter of the circle passing through the centre c and cutting qvq' in v; and let pt be the tangent to the circle at p.

Then pcp' projects into PCP', passing through C, the centre of the ellipse; pt projects into PT, touching the ellipse at P (Art. 48); qvq' projects into QVQ', parallel to PT. (Art. 44.)

Now, by a law of projection, (Art. 46.)

$$QV : Q'V = qv : q'v.$$

And, by a property of the circle,

since pp' is perpendicular to pt, and therefore to qq',

$$qv = q'v. \qquad \text{(Eucl. III. 3.)}$$

Hence $QV = Q'V$.

Thus the locus of the middle points of all chords in the ellipse parallel to QQ' is a straight line passing through the centre.

Also, the tangents at the ends of this line are parallel to the chords.

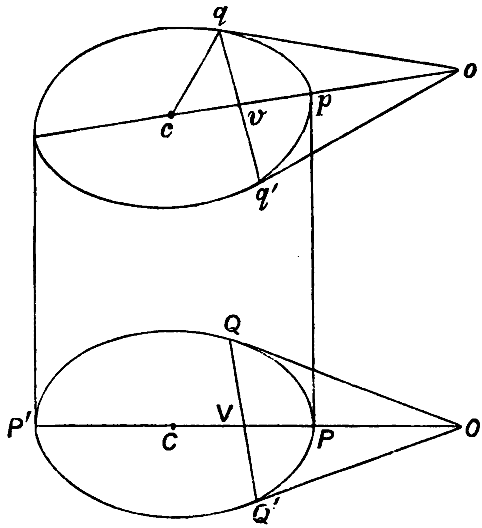
DEF. The straight line, which passes through the middle points of a system of parallel chords, is called a *Diameter*.

DEF. The segments of the chords bisected by any diameter are called the *Ordinates* of that diameter.

Note.—All diameters of an ellipse pass through the centre; and, conversely, any line passing through the centre may be considered a diameter.

In the propositions that follow, the word *Diameter* is restricted to that portion of a diameter which falls within the ellipse; thus we call PP' a diameter, and CP a semi-diameter.

88. *The tangents at the ends of a chord meet on the diameter bisecting that chord.*



Making the same construction as before,

let the tangent at q meet cp produced in o. Join oq'.

Then, since qq' is bisected by the perpendicular ov, $oq' = oq$,

and $\therefore oq'$ is a tangent to the circle. (Eucl. III. 37

Now qo projects into QO, a tangent at Q;

co projects into CO;

qq' projects into QQ'.

Hence, since O and Q' are the projections of o and q', OQ' is the projection of oq', and is therefore a tangent at Q'.

Thus the tangents at the extremities of the chord QQ' meet the diameter bisecting the chord in the same point.

89. *If QV be an ordinate of the diameter PP', and if the tangent at Q meet CP produced in O, then $CV.CO = CP^2$.*

Making the same construction as before, and joining cq, cqo is a right angle, and qv is perpendicular to co;

$$\therefore cv : cq = cq : co, \qquad \text{(Eucl. VI. 8.)}$$

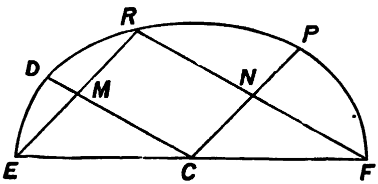
$$\text{or, } cv : cp = cp : co;$$

and therefore, by a law of projection, (Art. 46.)

$$CV : CP = CP : CO;$$

$$\text{or, } CV.CO = CP^2.$$

90. *If CP bisects chords parallel to CD, then CD bisects chords parallel to CP.*



Let CP bisect any chord RNF parallel to CD in N.
Draw the diameter FCE through F.
Join RE, cutting CD in M.

Then, since $FC = CE$, and $FN = NR$,

$$\therefore RE \text{ is parallel to } PC. \qquad \text{(Eucl. VI. 2.)}$$

And since CM is parallel to FR, and $FC = CE$,

$$\therefore RM = ME. \qquad \text{(Eucl. VI. 2.)}$$

Thus CD bisects chords parallel to CP.

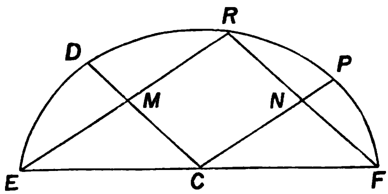
Conjugate Diameters.

91. DEF. Two diameters, each of which bisects the chords parallel to the other, are called *Conjugate Diameters.*

If two diameters of a circle are so related that each bisects the chords parallel to the other, such diameters must be at right angles to each other. Hence it follows that such diameters of a circle project into conjugate diameters of an ellipse.

92. DEF. The chords which join the ends of any diameter to any point on the curve are called *Supplemental Chords.*

The diameters drawn parallel to Supplemental Chords are Conjugate.



Let ER, FR be supplemental chords.

Take M and N, the middle points of ER, FR.

Draw CMD, CNP, to meet the ellipse in D, P.

Then, since CD bisects ER and EF.

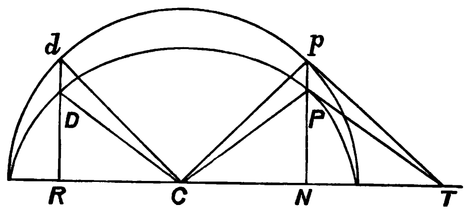
$\therefore$ CD is parallel to FR. (Eucl. VI. 2.)

Similarly CP is parallel to ER.

And CP, CD are conjugate semi-diameters,

since each bisects a chord parallel to the other.

93. *If CP, CD be conjugate semi-diameters, and the ordinates PN, DR meet the auxiliary circle in p, d, then dCp is a right angle.*



Let the tangents to the ellipse at P, and the auxiliary circle at p, meet the major axis produced in T. (Art. 76.)

Then, since PT is parallel to CD,

$PN : NT = DR : RC$ (1). (Eucl. VI. 4.)

Also, $pN : PN = dR : DR$ (2). (Art. 59.)

Compounding (1) and (2).

$pN : NT = dR : RC$;

$\therefore$ dRc, pNT are similar triangles; (Eucl. VI. 6.)

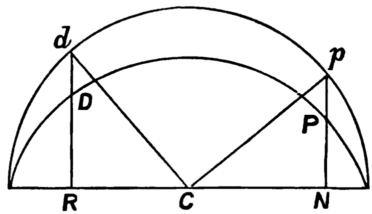
$\therefore$ dC is parallel to pT;

$\therefore$ $\angle dCp = \angle CpT$

$=$ a right angle.

94. *To prove that $PN : CR = CB : CA$, and that*

$$DR : CN = CB : CA.$$



The same construction being made as before, we can show that the triangles dRC, CNp are equal in all respects.

For, since $\angle dCp$ is a right angle,

$$\therefore \angle dCR + \angle pCN = \text{a right angle}$$
$$= \angle CpN + \angle pCN;$$
$$\therefore \angle dCR = \angle CpN.$$

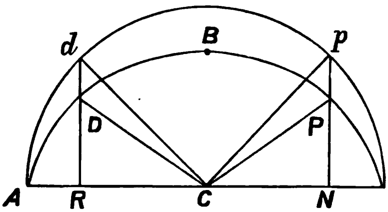
Also the right angles dRC, CNp are equal; and $Cd = pC$;

$\therefore$ the triangles dRC, CNp are equal in all respects, and $CR = pN$, and $dR = CN$. (Eucl. I. 26.)

Hence $PN : CR = PN : pN$
$= CB : CA$. (Art. 59.)

And $DR : CN = DR : dR$
$= CB : CA$. (Art. 59.)

95. *To prove that* $CP^2 + CD^2 = CA^2 + CB^2$.



The same construction being made,

$$DR : PN = dR : pN; \qquad \text{(Art. 59.)}$$

$$\therefore DR^2 : PN^2 = dR^2 : pN^2;$$

$$\begin{aligned}\therefore DR^2 + PN^2 : PN^2 &= dR^2 + pN^2 : pN^2 \\ &= dR^2 + CR^2 : pN^2 \\ &= Cd^2 : pN^2 \\ &= CA^2 : pN^2;\end{aligned}$$

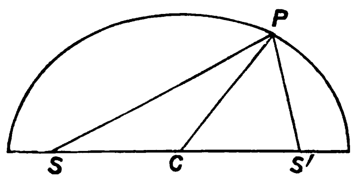
$$\begin{aligned}\therefore DR^2 + PN^2 : CA^2 &= PN^2 : pN^2 \\ &= CB^2 : CA^2; \qquad \text{(Art. 59.)}\end{aligned}$$

$$\therefore DR^2 + PN^2 = CB^2.$$

Then

$$\begin{aligned}CP^2 + CD^2 &= CN^2 + PN^2 + CR^2 + DR^2 \\ &= dR^2 + PN^2 + CR^2 + DR^2 \quad \text{(Art. 94.)} \\ &= (dR^2 + CR^2) + (PN^2 + DR^2) \\ &= dC^2 + CB^2 \\ &= CA^2 + CB^2.\end{aligned}$$

96. *To show that $SP \, . \, S'P = CD^2$.*



Since C is the middle point of SS',

$\therefore SP^2 + S'P^2 = 2CP^2 + 2CS^2$. (1) (Art. 8 (8).)

Also, since $SP + S'P = 2CA$,

$\therefore SP^2 + S'P^2 + 2SP \, . \, S'P = 4CA^2$ (2) (Eucl. II. 4.)

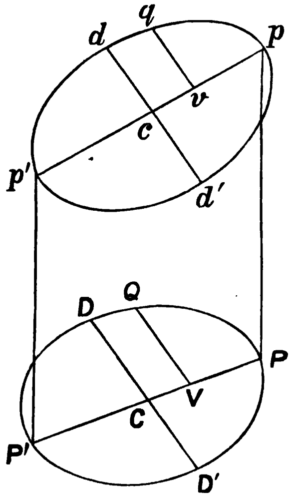
Subtracting (1) from (2),

$$2SP \, . \, S'P = 4CA^2 - 2CP^2 - 2CS^2;$$

$$\begin{aligned} \therefore SP \, . \, S'P &= 2CA^2 - CP^2 - CS^2 \\ &= CA^2 + CA^2 - CS^2 - CP^2 \\ &= CA^2 + CB^2 - CP^2 \quad \text{(Art. 66.)} \\ &= CP^2 + CD^2 - CP^2 \quad \text{(Art. 95.)} \\ &= CD^2. \end{aligned}$$

97. *If QV be an ordinate of the diameter PCP', and DCD' be the diameter parallel to QV, then*

$$PV \,.\, VP' : QV^2 = CP^2 : CD^2.$$



Let pcp', dcd', diameters of a circle at right angles to each other, project into the conjugate diameters PCP', DCD' of an ellipse. Also let qv, an ordinate of the circle parallel to cd, project into QV, parallel to CD, in the ellipse.

Then $PV : PC = pv : pc$, (Art. 46.)

and $VP' : P'C = vp' : p'c$; (Art. 46.)

$\therefore$ observing that $PC = P'C$ and $pc = p'c$,

$PV.VP' : PC^2 = pv \,.\, vp' : pc^2$

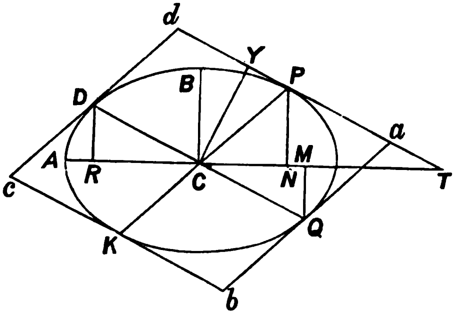
$= qv^2 : cd^2$ (Eucl. III. 35.)

$= QV^2 : CD^2$ (Art. 45.)

or, $PV.VP' : QV^2 = CP^2 : CD^2$.

98. *The area of the parallelogram formed by drawing tangents at the extremities of conjugate diameters is constant, and is equal to* $4CA \,.\, CB$.

Let *abcd* be the parallelogram formed by tangents parallel to the conjugate diameters PK, DQ.



Draw CY perpendicular to the tangent at P, and let this tangent produced meet the major axis in T.

Draw the ordinates DR, PN, QM; of these $DR = QM$.

Then, since CYT, QMC are similar triangles,

$$CT : CY = CQ : QM;$$

also, $CN : CA = DR : CB$; (Art. 94.)

And, compounding those proportions,

$$CT \,.\, CN : CY \,.\, CA = CQ : CB;$$

or, $CA^2 : CY.CA = CQ : CB$; (Art. 75.)

or, $CA : CY = CD : CB$;

and $\therefore$ $CY \,.\, CD = CA \,.\, CB$.

Now $CY.CD =$ area of parallelogram $dDCP$, which is one-fourth of parallelogram *abcd*;

$\therefore$ area of $abcd = 4CA \,.\, CB$.

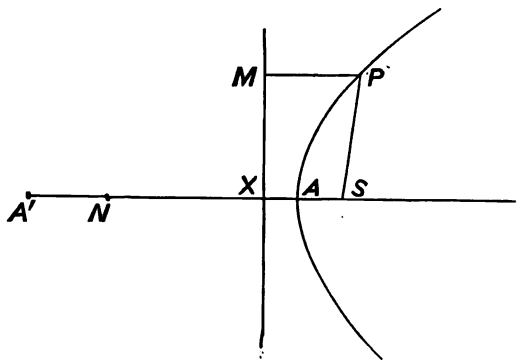
Cor. If PF be the perpendicular from P on DQ,

$$PF = CY, \text{ and } \therefore PF \,.\, CD = CA \,.\, CB.$$

CHAPTER V.

The Hyperbola.

99. Def. If a point P move in such a way that its distance SP from a fixed point S is in a constant ratio, greater than unity, to its perpendicular distance PM from a fixed straight line MX, the curve traced out by P is called a *Hyperbola.*



The point S is called a *Focus;* the line MX is called a *Directrix.*

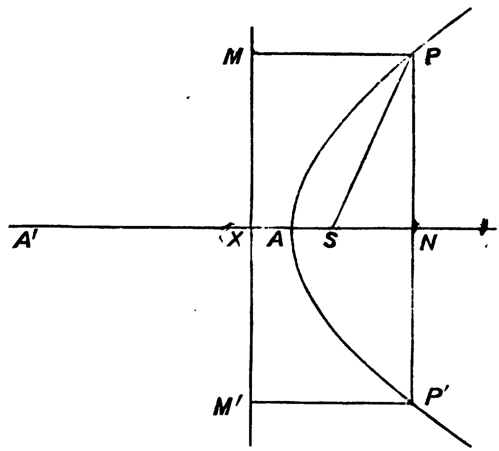
Through S draw SX perpendicular to the directrix; the line SX produced both ways is called the *Axis.*

The constant ratio of $SP : PM$ is called the *Eccentricity.*

In SX take a point A, such that $SA : AX = SP : PM$; then A is a point in the hyperbola: this point is called a *Vertex.*

In SX produced take a point A', such that $SA' : A'X = SP : PM$; then A' is a point in the hyperbola: this point is likewise called a *Vertex*.

100. *To construct the curve by the determination of successive points in it.*



Through N, any point in AS, or AS produced, draw a straight line perpendicular to the axis, and take SP of such a length that $SP : NX = SA : AX$.

With S as centre, and radius SP, describe a circle cutting the straight line in P, P'. Draw PM, $P'M'$ perpendiculars to the directrix.

$$\text{Then } SP : PM = SP : NX$$
$$= SA : AX;$$

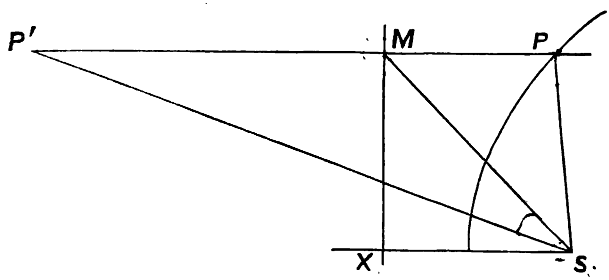
$\therefore$ P is a point on the curve.

Similarly P' is a point on the curve.

In this way, by taking N at other points in the line AS or AS produced, we might determine any number of points in the curve. And since $PN=P'N$, the curve will be symmetrical with regard to the axis.

101. As N is taken further from A, NX increases, and therefore SP increases. Hence the curve recedes from the directrix continuously, and for all positions of P, except when it coincides with A, SP is greater than AS.

102. The condition that SP is greater than PM can be satisfied on whichever side of MX the point P is taken, and hence we might expect to find a portion of the curve on the other side of the directrix. We proceed to show that for any point P on the curve, described as in Art. 100, there is a corresponding point P' on another branch of the curve on the opposite side of the directrix.



Join SM, and draw SP' making the angle MSP' equal to the angle MSP, and let SP' meet PM produced in P'.

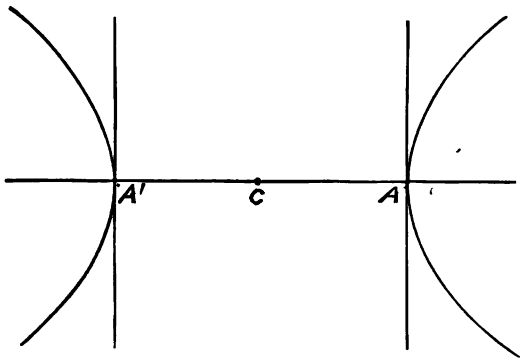
Then since $\angle MSP = \angle MSP'$,

$\therefore SP' : SP = P'M : PM;$ (Eucl. VI. 3.)

$\therefore SP' : P'M = SP : PM =$ the eccentricity;

$\therefore P'$ is a point in the hyperbola.

103. *General description of the form of the curve.*

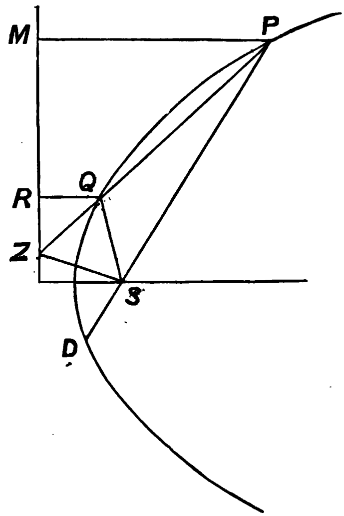


The Hyperbola consists of two opposite similar and infinite branches, lying outside lines drawn through the vertices A, A' at right angles to the axis.

If AA' be bisected in C, C is called the *Centre* of the hyperbola.

And AA' is called the *Transverse Axis*.

104. *If the chord PQ meet the directrix in Z, ZS will bisect QSD, the exterior angle between PS and QS.*



Draw PM and QR perpendicular to the directrix.

Then $PS : QS = PM : QR$, from the definition of the curve,

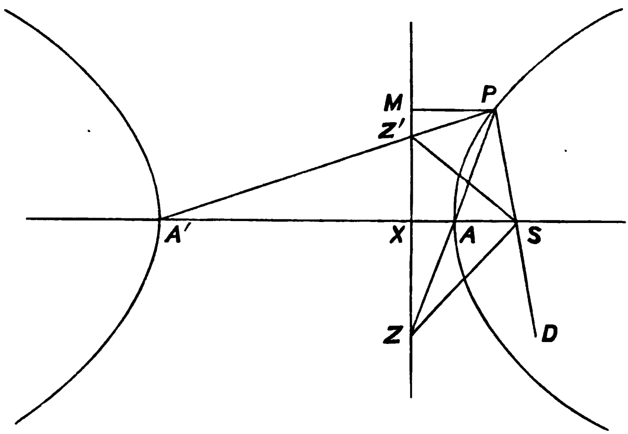
$= PZ : QZ$, (Eucl. VI. 2.)

and $\therefore ZS$ bisects $\angle QSD$. (Eucl. VI. A.)

Ex. 1. Show that the same result follows when P and Q are on the other branch of the curve.

Ex. 2. Show that, when P and Q are on different branches of the curve, ZS will bisect the interior angle between PS and QS.

105. *If two chords passing through the vertices A, A' and any point P in the curve, meet the directrix in ZZ', then $ZX \cdot Z'X = SX^2$.*



Let PA, $A'P$ be the two chords.

Produce PS to D; draw PM perpendicular to directrix.

Then $PS : AS = PM : AX$, from the definition of the curve,

$= PZ : AZ$, (Eucl. VI. 2.)

and $\therefore ZS$ bisects $\angle ASD$. (Eucl. VI. A.)

And $PS : A'S = PM : A'X$, from the definition of the curve,

$= PZ' : A'Z'$, (Eucl. VI. 4.)

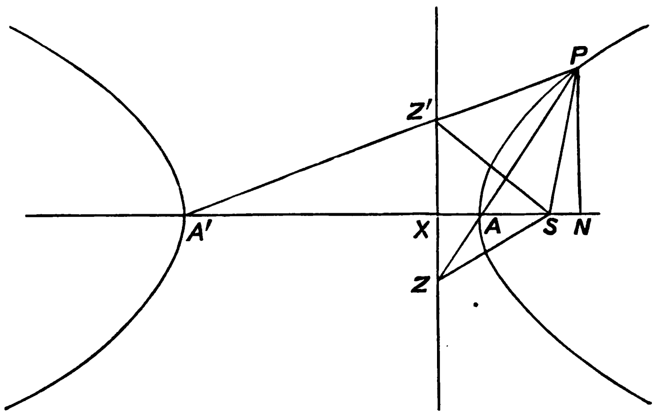
and $\therefore Z'S$ bisects $\angle A'SP$. (Eucl. VI. 3.)

Hence ZSZ' is a right angle, (Art. 8. (4).)

and $\therefore ZX \cdot Z'X = SX^2$. (Eucl. VI. 8.)

Ex. Show that the same result follows when P is on the other branch of the curve.

106. *If PN be the ordinate of a point P in the curve, then $PN^2 : AN \,.\, A'N$ is a constant ratio.*



Let the chords PA, PA' meet the directrix in Z, Z'.

By similar triangles PAN, ZAX

$$PN : AN = ZX : AX; \qquad (1)$$

and by similar triangles $PA'N$, $Z'A'X$

$$PN : A'N = Z'X : A'X. \qquad (2)$$

Compounding (1) and (2),

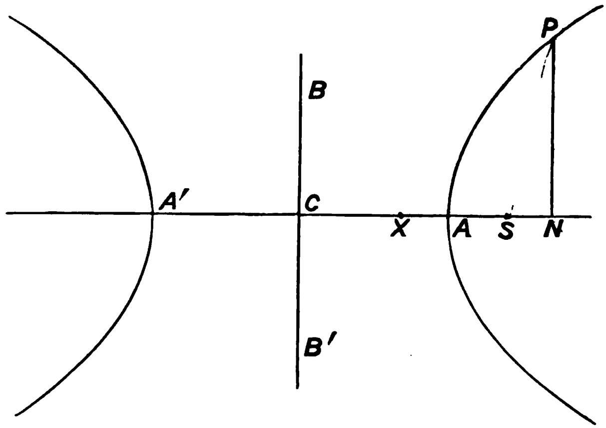
$$PN^2 : AN \,.\, A'N = ZX \,.\, Z'X : AX \,.\, A'X$$

$$= SX^2 : AX \,.\, A'X \qquad \text{(Art. 105.)}$$

$$= \text{a constant ratio for all positions of } P.$$

Ex. Show that the same result follows when P is on the other branch of the curve.

107. *The Conjugate Axis.*



From C draw CB perpendicular to AA', and of such a length that $CB^2 : CA^2 = SX^2 : AX . A'X$.

Then $PN^2 : AN . A'N = CB^2 : CA^2$. (Art. 106.)

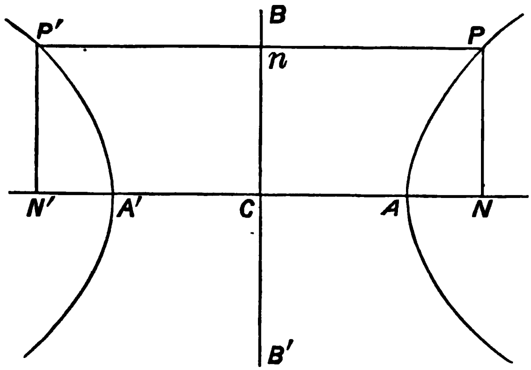
Now $AN . A'N + CA^2 = CN^2$; (Eucl. II. 6.)

$\therefore PN^2 : CN^2 - CA^2 = CB^2 : CA^2$.

Produce BC to B', making $CB' = CB$;

then BB' is called the *Conjugate Axis* of the hyperbola.

108. *The Hyperbola is symmetrical with respect to the conjugate axis.*



Take $A'N' = AN$; draw the ordinates PN, $P'N'$; and join PP', cutting BB', produced if necessary, in n.

Then $PN^2 : AN \,.\, A'N = CB^2 : CA^2$, (Art. 106.)

and $P'N'^2 : A'N' \,.\, AN' = CB^2 : CA^2$, (Art. 106, Ex.)

and $\therefore$ since $A'N' = AN$, and $AN' = A'N$,

$$P'N'^2 = PN^2,$$

and $\therefore P'N' = PN$.

Hence PP' is parallel and equal to NN'; (Eucl. I. 33.)

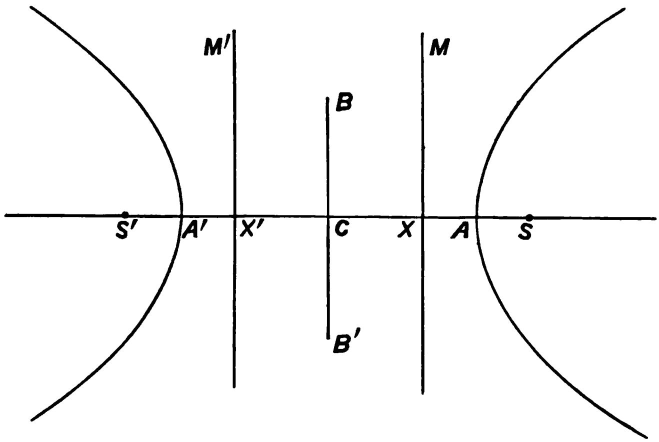
and $\therefore BB'$ is perpendicular to PP';

and, since NN' is bisected in C,

PP' is bisected in n. (Art. 8 (1).)

Thus the curve is symmetrical with respect to the conjugate axis.

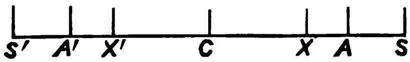
109. *The Hyperbola has a second focus and a second directrix.*



In AA' produced take $A'S' = AS$, and in AA' take $A'X' = AX$, and draw $M'X'$ perpendicular to AA'.

Then S' is a second focus, and $M'X'$ is a second directrix. For, since the curve has been proved to be symmetrical with regard to each axis, it is clear that the curve may be described by means of the focus S' and the directrix $M'X'$ exactly in the same way as by means of the focus S and the directrix MX.

110. *To show that* (1) $CS : CA =$ *the eccentricity ;*
(2) $CA : CX =$ *the eccentricity ;*
(3) $CS \,.\, CX = CA^2$.



(1.) Since $SA' : A'X = SA : AX$,

$$\therefore SA' : SA = A'X : AX;$$
$$\therefore SA' + SA : SA = A'X + AX : AX;$$
$$\therefore SA' + S'A' : SA = A'X + AX : AX;$$
$$\therefore SS' : SA = AA' : AX;$$
$$\therefore SS' : AA' = SA : AX;$$
$$\therefore CS : CA = SA : AX;$$

that is, $CS : CA =$ the eccentricity.

(2.) Since $SA' : A'X = SA : AX$,

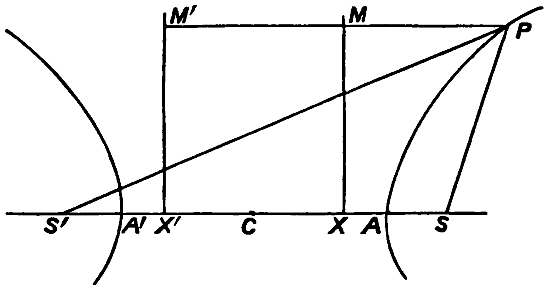
$$\therefore SA' : SA = A'X : AX;$$
$$\therefore SA' - SA : SA = A'X - AX : AX;$$
$$\therefore SA' - SA : SA = A'X - A'X' : AX;$$
$$\therefore AA' : SA = XX' : AX;$$
$$\therefore AA' : XX' = SA : AX;$$
$$\therefore CA : CX = SA : AX;$$

this is, $CA : CX =$ the eccentricity.

(3.) Since $CS : CA$ and $CA : CX$ are ratios each of which is equal to the eccentricity,

$$\therefore CS : CA = CA : CX;$$
$$\text{and } \therefore CS \,.\, CX = CA^2.$$

111. *The difference of the focal distances of any point on the hyperbola is equal to the transverse axis.*



Draw PMM' to meet the directrices at right angles.

Then $S'P : SP = PM' : PM$;

$\therefore S'P - SP : SP = PM' - PM : PM$;

$\therefore S'P - SP : PM' - PM = SP : PM$;

$\therefore S'P - SP : XX' = SP : PM$

$= CA : CX$ (Art. 110.)

$= 2CA : 2CX$

$= AA' : XX'$;

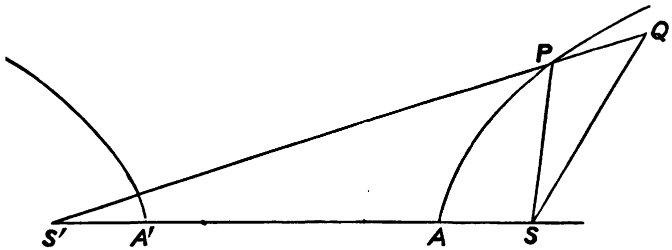
$\therefore S'P - SP = AA'$.

Note.—If the point P be on the other branch of the hyperbola,
$SP - S'P = AA'$.

112. *The difference of the focal distances of a point within the hyperbola is greater than the transverse axis.*

Let Q be any point within either branch of the hyperbola.

Join $S'Q$, SQ, and let $S'Q$ cut the hyperbola in P. Join SP.



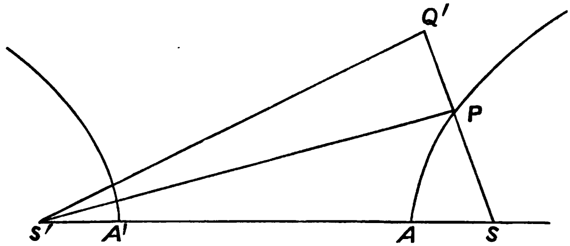
Then, since SQ is less than $PQ+SP$, (Eucl. I. 20.)

$\therefore$ $S'Q-SQ$ is greater than $S'Q-(PQ+SP)$;

$\therefore$ $S'Q-SQ$ is greater than $S'P-SP$;

$\therefore$ $S'Q-SQ$ is greater than AA'.

113. *The difference of the focal distances of a point without the hyperbola is less than the transverse axis.*



Let Q' be any point without the hyperbola.

Join $S'Q'$, $Q'S$, and let $Q'S$ cut the hyperbola in P. Join $S'P$.

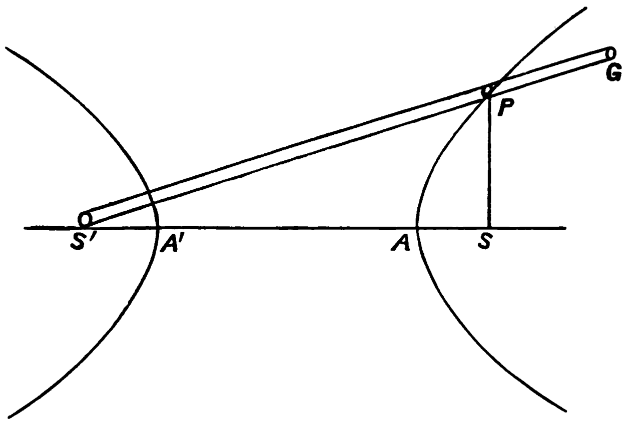
Then, since $S'Q'$ is less than $S'P+Q'P$, (Eucl. I. 20.)

$\therefore$ $S'Q'-SQ'$ is less than $S'P+Q'P-SQ'$;

$\therefore$ $S'Q'-SQ'$ is less than $S'P-SP$;

$\therefore$ $S'Q'-SQ'$ is less than AA'.

114. *The mechanical construction of a hyperbola.*



Take two points S, S' for foci, and let a rod $S'PG$ revolve about S'. At S fasten one end of a string of a length such that

Length of rod − length of string = the transverse axis of the proposed hyperbola.

Let the other end of this string be fastened at G, and let a pencil P, sliding along the rod, keep the string tight against the rod.

Then, as the rod revolves round S', P will trace out one branch of the hyperbola; for

$$\begin{aligned} S'P - SP &= (S'P + PG) - (SP + PG) \\ &= \text{length of rod} - \text{length of string} \\ &= AA'. \end{aligned}$$

The other branch of the curve may be traced by making S the point round which the rod revolves, and fastening the string at S' and at the end of the rod remote from S.

115. *To prove that* $CB^2 = CS^2 - CA^2 = AS \,.\, A'S$.

Since $CS : CA = SA : AX$, (Art. 110.)

$\therefore\ CS + CA : CA = SA + AX : AX$

$= SX : AX$. (1.)

Also, $CS : CA = SA' : A'X$; (Art. 110.)

$\therefore\ CS - CA : CA = SA' - A'X : A'X$

$= SX : A'X$. (2.)

Compounding (1) and (2),

$CS^2 - CA^2 : CA^2 = SX^2 : AX \,.\, A'X$.

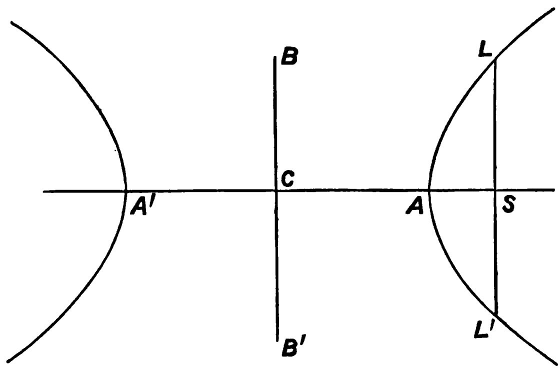
But $CB^2 : CA^2 = SX^2 : AX \,.\, A'X$; (Art. 107.)

$\therefore\ CB^2 = CS^2 - CA^2$.

$= (CS - CA)\,(CS + CA)$ (Art. 9.

$= AS \,.\, A'S$.

116. *To find an expression for the Latus Rectum.*



DEF. The double ordinate LL' through either focus is called the *Latus Rectum.*

$$\text{Now } LS^2 : AS.A'S = CB^2 : CA^2; \qquad \text{(Art. 107.)}$$

$$\therefore LS^2 : CB^2 = CB^2 : CA^2; \qquad \text{(Art. 115.)}$$

$$\therefore LS : CB = CB : CA;$$

$$\therefore LS.CA = CB^2,$$

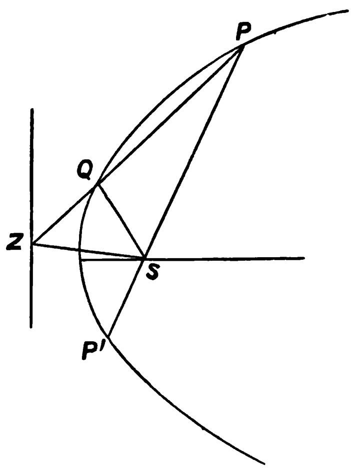
$$\text{or } LS = \frac{CB^2}{CA}.$$

Hence *The Latus Rectum* $= \dfrac{2CB^2}{CA}$.

The Tangent.

117. A secant to the hyperbola cuts the curve in two points only. The proof of this is similar to that which is given in Art. 21 for the same property in the parabola. And, as in the parabola, a tangent to a hyperbola is the ultimate position of a secant when the two points in which the secant cuts the curve have become coincident. (*See* Art. 22.)

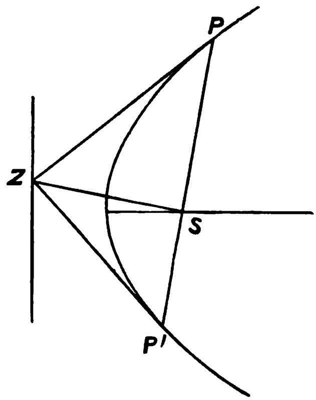
118. *If the tangent at P meets the directrix in Z, then PSZ is a right angle.*



Let the chord PQ meet the directrix in Z.

Produce PS to meet the curve in P'; then ZS bisects $\angle QSP'$. (Art. 104.)

Now let Q move up to P; then, when Q coincides with P, the line QS coincides with PS, the angle QSP' becomes equal to two right angles, and ZS is perpendicular to PP'.

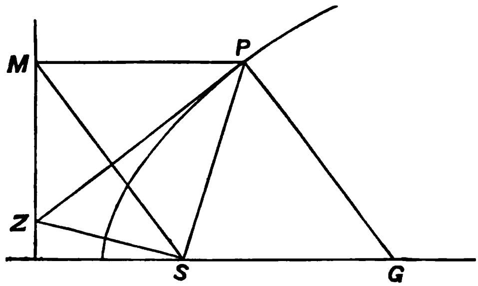


In this position ZP is the tangent at P.

Hence, if we join ZP', the line ZP' will be the tangent at P', and we see that the tangents at the extremities of a focal chord intersect in the directrix.

Note.—In the same way it may be shown that if the tangent at P meet the other directrix in Z', the angle $Z'S'P$ is a right angle.

119. *If the normal to the hyperbola at P meets the transverse axis in G, then $SG : SP =$ the eccentricity.*



Let PZ be the tangent at P, meeting the directrix in Z. Draw PM perpendicular to the directrix.

Since PMZ and PSZ are right angles, a circle described on PZ as diameter will pass through M and S, and PG, which is perpendicular to PZ, will touch the circle at P. (Eucl. III. 16.)

Hence $\angle SPG = \angle PMS$ in the alternate segment; (Eucl. III. 32.)

and $\angle PSG = \angle MPS$, since PM is parallel to SG;

$\therefore SPG$, SMP are similar triangles,

and $\therefore SG : SP = SP : PM$,

that is, $SG : SP =$ the eccentricity.

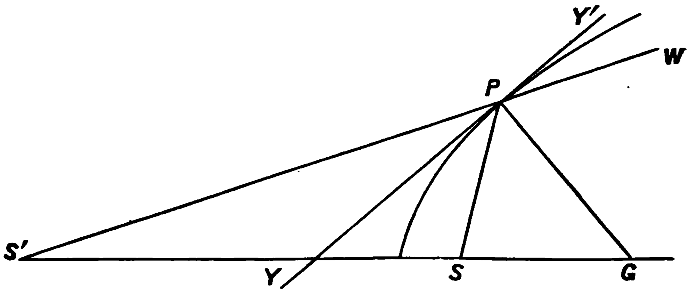
Similarly, by producing PZ and PM to meet the other directrix, it may be shown that

$$S'G : S'P = \text{the eccentricity.}$$

Hence it follows that

$$S'G : SG = S'P : SP.$$

120. *The tangent and the normal at P respectively bisect the interior and exterior angles between the focal distances of P.*



Produce $S'P$ to W.

Let YPY' be the tangent at P, and PG the normal at P.

Then $S'G : SG = S'P : SP$, (Art. 119.)

and $\therefore$ PG bisects the angle SPW, (Eucl. VI. A.)

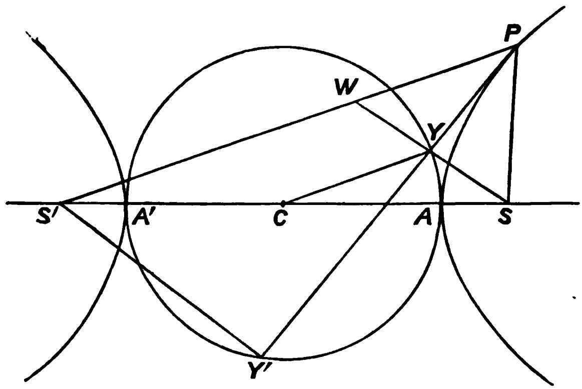
that is, the normal bisects the exterior angle between the focal distances of P.

And PY is perpendicular to PG;

$\therefore$ PY bisects the angle SPS'; (Art. 8 (4).)

that is, the tangent bisects the angle between the focal distances of P.

121. *The perpendiculars from the foci on the tangent intersect the tangent in the circumference of a circle described on the transverse axis as diameter, which is called the auxiliary circle.*



Let SY, $S'Y'$ be perpendiculars on the tangent at P.
Let SY produced cut $S'P$ in W.

Then, since $\angle WPY = \angle SPY$, (Art. 120.)
and $\angle PYW = \angle PYS$, being right angles,

$\therefore PW = SP$; (Eucl. I. 26.)

and $\therefore S'W = S'P - PW$
$= S'P - SP$
$= 2CA$.

Now CY bisects $S'S$ and WS;

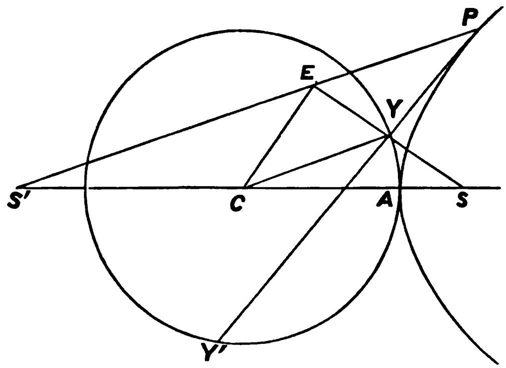
$\therefore CY$ is parallel to $S'W$; (Eucl. VI. 2.)

$\therefore S'W = 2CY$ (Eucl. VI. 4.)

$\therefore CA = CY$.

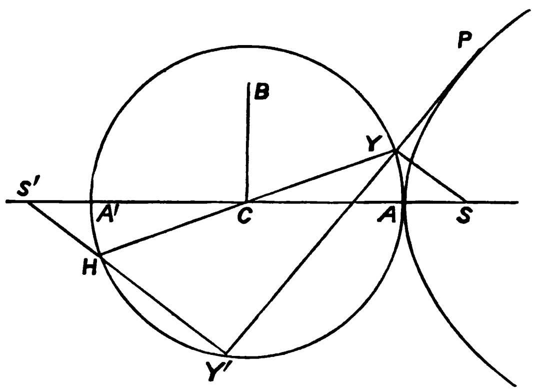
Hence Y is a point on the auxiliary circle.
Similarly Y' may be proved to be a point on that circle.

122. *If CE, drawn parallel to PY, meets $S'P$ in E, then $PE = CA$.*



Since $PECY$ is a parallelogram, $\therefore PE = CY = CA$.

123. *To show that $SY . S'Y' = CB^2$.*



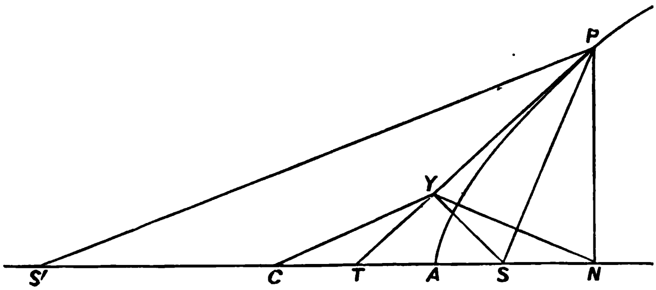
Let $S'Y'$ meet the auxiliary circle again in H.
Then, since $HY'Y$ is a right angle,
HY is a diameter passing through C;
and $\therefore CH = CY$.

But $S'C = SC$ and $\angle S'CH = \angle SCY$;

$$\therefore S'H = SY. \qquad \text{(Eucl. I. 4.)}$$

$$\begin{aligned} \text{And } SY . S'Y' &= S'H . S'Y' \\ &= S'A' . S'A \qquad \text{(Eucl. III. 36.)} \\ &= AS . AS' \\ &= CB^2. \qquad \text{(Art. 115.)} \end{aligned}$$

124. *If the tangent at P meets the transverse axis in T, then $CN . CT = CA^2$.*



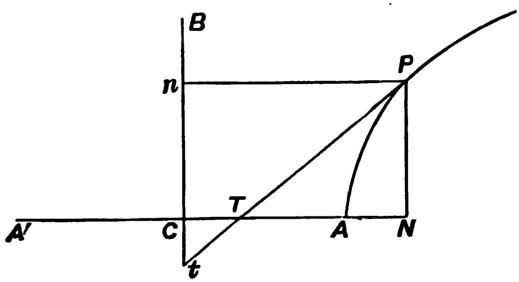
Draw SY perpendicular to PT.
Then, since PNS and PYS are right angles,
a circle can be described round $PNSY$;
and $\therefore \angle SNY = \angle SPY$ in the same segment.
Then we can show that CNY, CYT are similar triangles;
for they have a common angle at C,

$$\begin{aligned} \text{and } \angle CNY &= \angle SNY \\ &= \angle SPY \\ &= \angle S'PT \qquad \text{(Art. 120.)} \\ &= \angle CYT; \qquad \text{(Art. 121.)} \end{aligned}$$

$$\therefore CN : CY = CY : CT;$$

$$\begin{aligned} \therefore CN . CT &= CY^2 \\ &= CA^2 \qquad \text{(Art. 121.)} \end{aligned}$$

125. *If the tangent at P meets the conjugate axis in t, and Pn be drawn perpendicular to the conjugate axis,* $Cn \,.\, Ct = CB^2$.



Since $Ct : CT = PN : NT$, by similar triangles CtT, NPT, and $Cn : CN = PN : CN$, since $PNCn$ is a rectangle,

$$\therefore Cn \,.\, Ct : CN \,.\, CT = PN^2 : CN \,.\, NT;$$

$$\therefore Cn \,.\, Ct : CA^2 = PN^2 : CN^2 - CN \,.\, CT \quad \text{(Eucl. II. 2.)}$$

$$= PN^2 : CN^2 - CA^2 \quad \text{(Art. 124.)}$$

$$= CB^2 : CA^2; \quad \text{(Art. 107.)}$$

$$\therefore Cn \,.\, Ct = CB^2.$$

126. *If PG, the normal at P, meets a line, drawn through C parallel to the tangent at P, in F; then shall* $PF \,.\, PG = CB^2$. (See diagram in Art. 127.)

Let the ordinate PN produced meet CF produced in K.

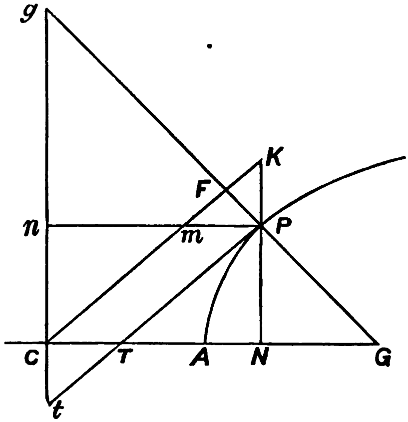
Then, since the angles at F and N are right angles, a circle can be described round $KFNG$.

$$\text{And } PF \,.\, PG = PN \,.\, PK \quad \text{(Eucl. III. 35.)}$$

$$= Cn \,.\, Ct$$

$$= CB^2. \quad \text{(Art. 125.)}$$

127. *If PG meets the conjugate axis in g, then* $PF.Pg = CA^2$.



Let Pn, drawn perpendicular to the conjugate axis, cut CF in m. Then, since the angles at n and F are right angles, a circle can be described round $mngF$.

$$\therefore PF.Pg = Pn.Pm \qquad \text{(Eucl. III. 36.)}$$
$$= CN.CT = CA^2. \qquad \text{(Art. 124.)}$$

128. *To show that* $GN : CN = CB^2 : CA^2$.

Since $PF.PG = CB^2$,

and $PF.Pg = CA^2$,

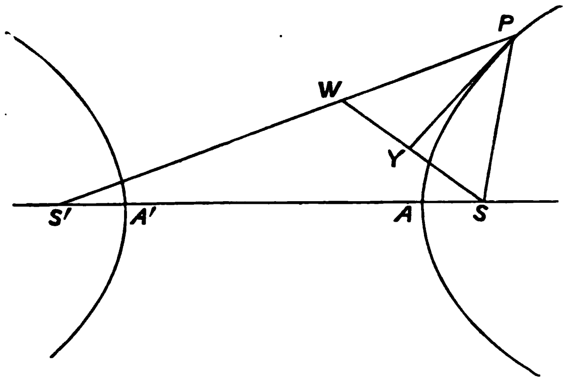
$$\therefore PG : Pg = CB^2 : CA^2;$$
$$\therefore GN : Pn = CB^2 : CA^2; \qquad \text{(Eucl. VI. 4.)}$$
$$\therefore GN : CN = CB^2 : CA^2.$$

129. *To show that* $CG : CN = CS^2 : CA^2$.

Since $GN : CN = CB^2 : CA^2$

$$\therefore GN + CN : CN = CB^2 + CA^2 : CA^2;$$
$$\therefore CG : CN = CS^2 : CA^2.$$

130. *To draw a tangent to the hyperbola from a given point on the curve.*



Let P be the given point.

Join SP, $S'P$, and in $S'P$ take $S'W = AA'$.

Join WS, and bisect the angle WPS by PY.

Then PY is a tangent at P. (Art. 120.)

Ex. 1. Show that the circle described on SP as diameter touches the auxiliary circle.

Ex. 2. If O be any point, exterior to the hyperbola and also to the auxiliary circle, and S be the focus nearest to O, show that the circle described on SO as diameter will cut the auxiliary circle.

131. *To draw a pair of tangents from a given point to the hyperbola.*

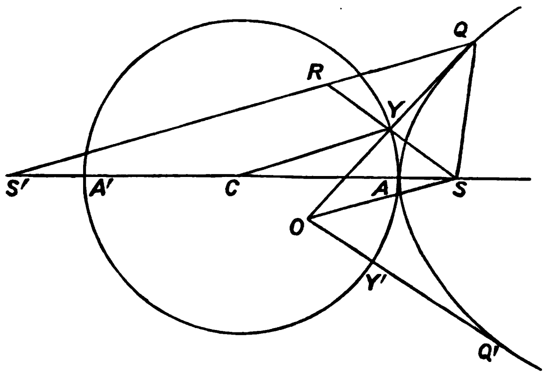
Let O be the given point, and S the focus nearest to O.

Join SO, and on it as diameter describe a circle cutting the auxiliary circle in Y, Y'.

Then OY, OY' produced will touch the hyperbola.

Produce SY to R, making $YR = SY$, and join $S'R$, and let it meet OY produced in Q.

Join CY; then CY bisects SS' and SR, and is therefore parallel to $S'RQ$, and $S'R = 2CY$.



Since OYS, the angle in a semi-circle, is a right angle,

$$\therefore \angle QYR = \angle QYS,$$

and since $RY = SY$, $\therefore SQ = RQ$. (Eucl. I. 4.)

$$\begin{aligned} \text{Then } S'Q - SQ &= S'Q - RQ \\ &= S'R \\ &= 2CY \\ &= AA'; \end{aligned}$$

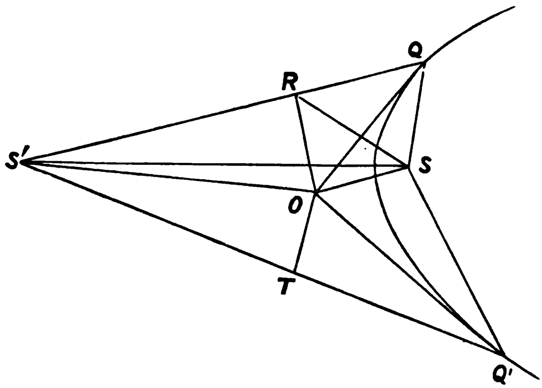
$\therefore Q$ is a point in the curve.

And since SY, perpendicular to OQ, meets it on the auxiliary circle,

$\therefore OQ$ is a tangent to the hyperbola. (Art. 121.)

Similarly, we might show that OY' produced is a tangent to the hyperbola at Q'.

132. *If OQ, OQ' be tangents drawn from O, meeting the hyperbola on the same branch,* $\angle OSQ = \angle OSQ'$.



In QS' take $QR = QS$,

and in $Q'S'$ take $Q'T = Q'S$.

Then, since OQ bisects $\angle RQS$, and $QR = QS$,

and OQ is common,

$\therefore \angle ORQ = \angle OSQ$,

and $OR = OS$.

Similarly, $\angle OTQ' = \angle OSQ'$,

and $OT = OS$;

and $\therefore OR = OT$.

Now $S'R = S'Q - QR = S'Q - QS = AA'$,

and $S'T = S'Q' - Q'T = S'Q' - Q'S = AA'$;

$$\therefore S'R = S'T,$$

and the triangles ORS', OTS' have all their sides equal;

$$\therefore \angle ORS' = \angle OTS',$$

$$\therefore \angle ORQ = \angle OTQ';$$

$$\text{and } \therefore \angle OSQ = \angle OSQ'.$$

133. *To show that $\angle QOS$ is the supplement of $\angle Q'OS'$.*

The six angles at O make up four right angles.

Of these $\angle ROQ + \angle QOS = 2 \angle QOS$,

$$\angle ROS' + \angle S'OT = 2 \angle S'OT,$$

$$\angle TOQ' + \angle Q'OS = 2 \angle Q'OT;$$

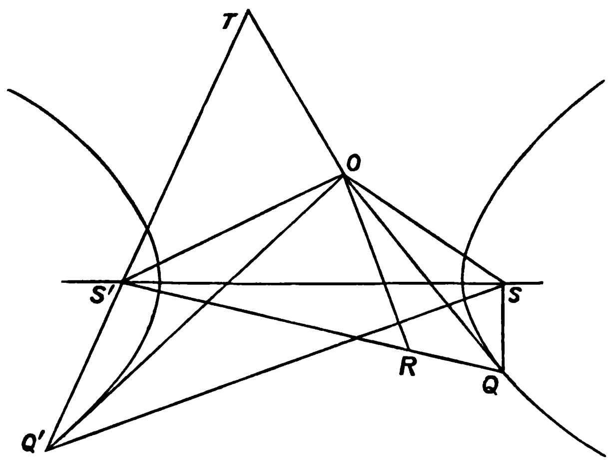
$\therefore 2 \angle QOS + 2 \angle S'OT + 2 \angle Q'OT =$ four right angles;

$\therefore \angle QOS + \angle S'OT + \angle Q'OT =$ two right angles;

$\therefore \angle QOS + \angle Q'OS' =$ two right angles;

$\therefore \angle QOS$ is the supplement of $\angle Q'OS'$.

134. *If OQ, OQ' be tangents drawn from O, meeting the hyperbola on opposite branches, $\angle\, OSQ$ is the supplement of $\angle\, OSQ'$.*



In QS' take $QR = QS$.

In $Q'S'$ produced take $Q'T = Q'S$.

Then, since OQ bisects $\angle\, RQS$, and $QR = QS$,

and OQ is common,

$\therefore\ \angle\, ORQ = \angle\, OSQ$,

and $OR = OS$.

Similarly $\angle\, OTQ' = \angle\, OSQ'$,

and $OT = OS$;

and $\therefore\ OR = OT$.

Now $S'R=S'Q-QR=S'Q-SQ=AA'$,

and $S'T=Q'T-S'Q'=SQ'-S'Q'=AA'$;

$\therefore$ $S'R=S'T$, and the triangles ORS', OTS' have all their sides equal;

$\therefore$ $\angle ORS'=\angle OTS'$;

$\therefore$ $\angle ORQ$ is the supplement of $\angle OTS'$;

$\therefore$ $\angle OSQ$ is the supplement of $\angle OTQ'$;

$\therefore$ $\angle OSQ$ is the supplement of $\angle OSQ'$.

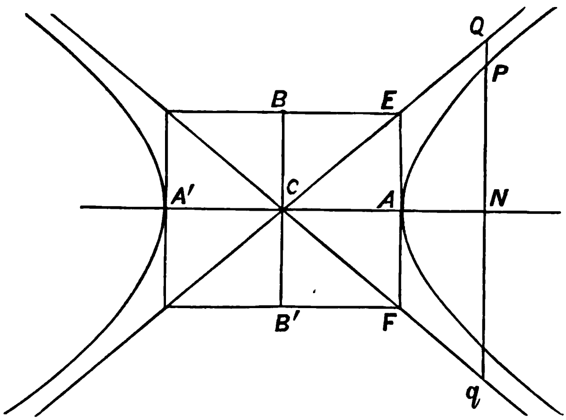
135. *To show that* $\angle SOQ=\angle S'OQ'$.

$$\begin{aligned}
\angle SOQ &= \frac{1}{2}\angle ROS \\
&= \frac{1}{2}(\angle SOS'-\angle S'OR) \\
&= \frac{1}{2}(\angle SOS'-\angle S'OT) \\
&= \frac{1}{2}(\angle SOQ'+\angle S'OQ'-\angle S'OT) \\
&= \frac{1}{2}(\angle Q'OT+S'OQ'-\angle S'OT) \\
&= \frac{1}{2}(\angle S'OQ'+\angle S'OQ') \\
&= \angle S'OQ'.
\end{aligned}$$

Asymptotes.

136. Def. — When a curve continually approaches nearer and nearer to a fixed straight line, but never meets the straight line at any finite distance, though the distance between the curve and the line becomes ultimately less than any finite length, the line is called a *Rectilinear Asymptote* to the curve. An Asymptote may be regarded as a tangent to the curve at a point which is infinitely distant.

137. *If PN, the ordinate of any point P on the curve, be produced to meet the diagonals of the rectangles, of which the semi-axes of the hyperbola are adjacent sides, in Q, q, then* $QP \cdot Pq = CB^2$; *and the diagonals are asymptotes of the hyperbola.*



Form a rectangle by drawing lines parallel to AA', BB' respectively through the points A, B, A', B'; and let the

diagonals CE, CF meet the ordinate PN produced both ways in Q, q.

Then, since CN bisects the angle QCq,

and CN is perpendicular to Qq,

$\therefore$ N is the middle point of Qq, (Eucl. I. 26.)

and $\therefore$ $QP \,.\, Pq = QN^2 - PN^2$. (Eucl. II. 5.)

Now $QN^2 : CN^2 = EA^2 : CA^2$ (Eucl. VI. 2.)

$= CB^2 : CA^2$

$= PN^2 : CN^2 - CA^2$; (Art. 107.)

$\therefore QN^2 : PN^2 = CN^2 : CN^2 - CA^2$;

$\therefore QN^2 - PN^2 : PN^2 = CA^2 : CN^2 - CA^2$; (Eucl. V. 17.)

$\therefore QN^2 - PN^2 : CA^2 = PN^2 : CN^2 - CA^2$

$= CB^2 : CA^2$;

$\therefore QN^2 - PN^2 = CB^2$;

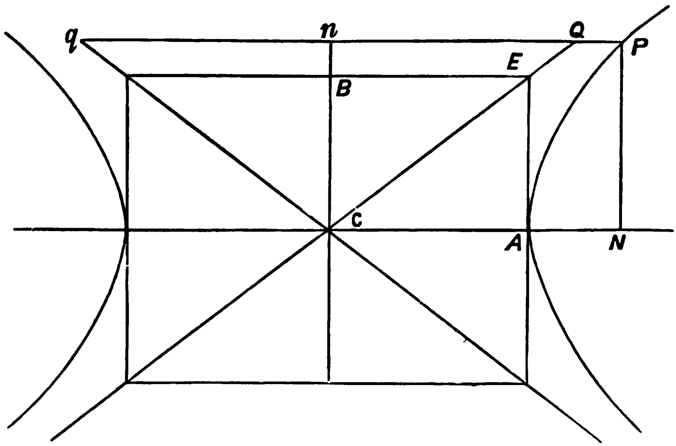
that is, $QP \,.\, Pq = CB^2$.

Now, as Pq increases continually, QP must decrease continually; and though QP can never be zero, yet it ultimately becomes less than any finite length; that is, CE is an asymptote to the hyperbola.

Similarly, CF is an asymptote.

Note.—The Asymptotes make equal angles with the Transverse Axis, and also with the Conjugate Axis.

138. *If Pn, the ordinate of P on the conjugate axis, meets the asymptotes in Q, q, then $QP \cdot Pq = CA^2$.*



Since n is the middle point of Qq,

$\therefore QP \cdot Pq = Pn^2 - Qn^2$. (Eucl. II. 6.)

Now $Qn^2 : Cn^2 = EB^2 : CB^2$ (Eucl. VI. 2.)

$= CA^2 : CB^2$

$= CN^2 - CA^2 : PN^2$. (Art. 107.)

And $Cn^2 = PN^2$;

$\therefore Qn^2 = CN^2 - CA^2$;

$\therefore QP \cdot Pq = CN^2 - (CN^2 - CA^2)$

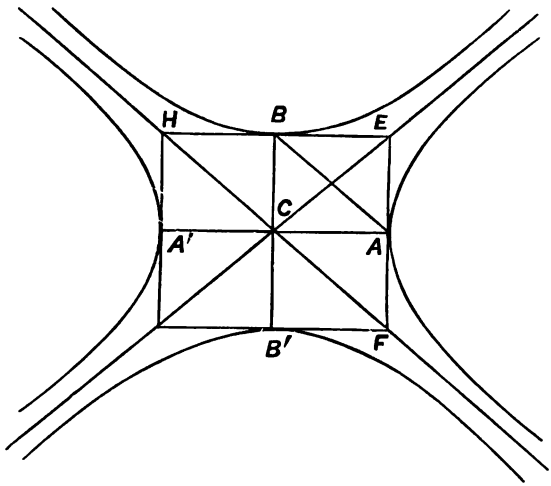
$= CA^2$.

The Conjugate Hyperbola.

139. Def.—A hyperbola which has BB' for its transverse axis and AA' for its conjugate axis is called the *Conjugate Hyperbola*.

The Original Hyperbola and its Conjugate Hyperbola are two distinct curves.

The Asymptotes of the Conjugate Hyperbola are the same as those of the Original Hyperbola.

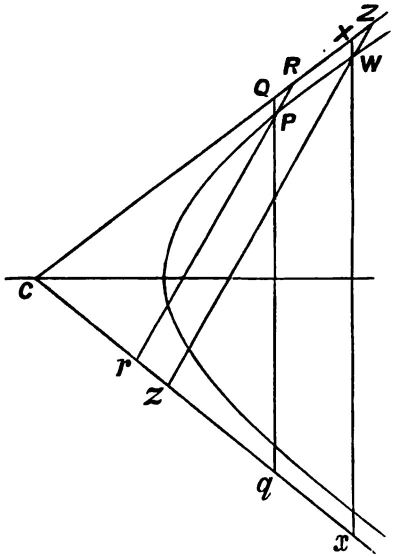


The line AB, joining the vertices of the Hyperbolas, is bisected by one asymptote and is parallel to the other; for it is bisected by CE, because the diagonals of the rectangle $BCAE$ bisect each other; and it is parallel to HF, because it bisects EH and EF.

All straight lines perpendicular to either axis, and terminated by the asymptotes, are bisected by the axis.

If the axes AA', BB' be equal, it is evident that CE bisects the angle BCA, and that CF bisects the angle $B'CA$, and therefore in this case the asymptotes are at right angles to each other, and the curve is called a *Rectangular Hyperbola.*

140. ***If any straight line whatever, Rr, terminated by the asymptotes and making a given angle with either asymptote, cuts the curve in P, the rectangle contained by the segments RP, Pr is invariable.***



Draw any other straight line Zz, terminated by the asymptotes, parallel to Rr, and let Zz cut the curve in W.

Through P, W draw Qq, Xx perpendicular to the axis.

Then, $RP : QP = ZW : XW$, (Eucl. VI. 4.)

and $Pr : Pq = Wz : Wx$; (Eucl. VI. 4.)

$\therefore RP.Pr : QP.Pq = ZW.Wz : XW.Wx$.

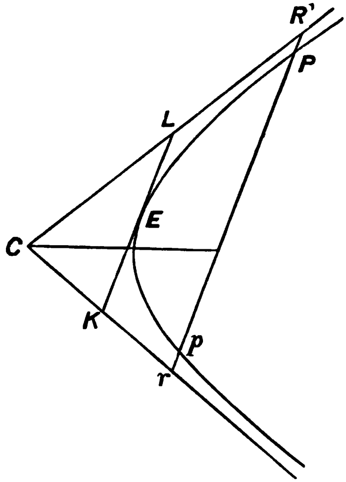
Now $QP.Pq = XW.Wx$,

each being equal to CB^2; (Art. 137.)

$\therefore RP.Pr = ZW.Wz$;

$\therefore$ the rectangle contained by the segments is invariable.

141. *If any straight line cuts the curve in P, p, and the asymptotes in R, r, then $RP = pr$.*



Let Rr move parallel to itself till the points P, p, in which it cuts the curve, coincide in E; then Rr becomes the tangent LEK.

Then, by the preceding proof,

$$RP \cdot Pr = LE \cdot EK.$$

Also, by the preceding proof

$$rp \cdot pR = KE \cdot EL;$$

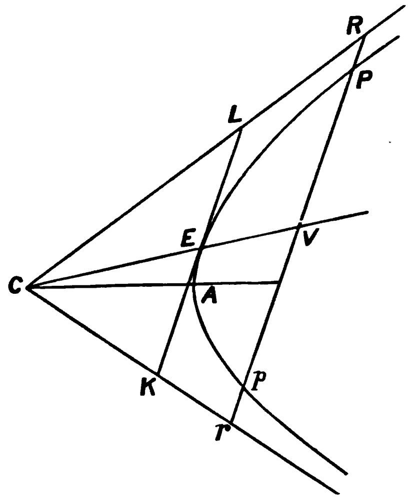
and $\therefore RP \cdot Pr = rp \cdot pR$,

or, $\quad RP \cdot (Pp + pr) = pr \cdot (RP + Pp)$;

and $\therefore RP \cdot Pp = pr \cdot Pp$;

and $\therefore RP = pr$.

142. ***The locus of the middle points of a system of parallel chords is a straight line passing through the centre of the hyperbola.***



Let Rr move parallel to itself, till the points P, p, in which it cuts the curve, coincide in E; then Rr becomes the tangent LEK.

Now, since RP is always equal to pr,

it follows that $LE = EK$.

Join CE, and produce it to meet Rr in V.

$$\text{Then } RV : LE = VC : EC$$
$$= Vr : EK;$$
$$\therefore RV = Vr;$$

and taking from each the equal lines RP, pr,

$$PV = Vp.$$

Thus the locus of the middle points of all chords in the hyperbola parallel to Rr is a straight line passing through the centre.

Also, the tangent at the point where this line cuts the curve is parallel to the chords.

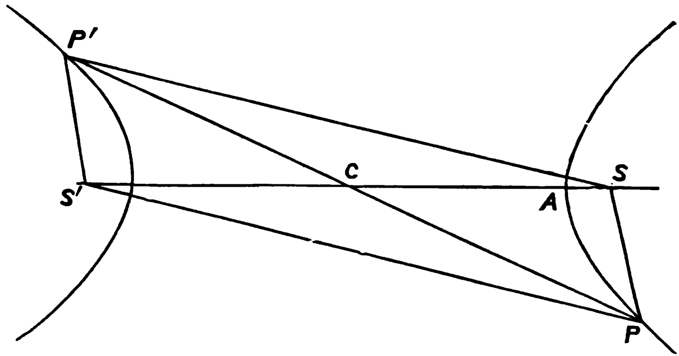
Def.—The straight line which passes through the middle points of a system of parallel chords is called a *Diameter*.

Def.—The segments of the parallel chords bisected by any diameter are called the *Ordinates* of that diameter.

Note.—All diameters of a hyperbola pass through the centre, and, conversely, every straight line passing through the centre may be considered a diameter.

In the propositions that follow, we apply the term *Diameter* to those parts of diameters which are intercepted by the branches of the original hyperbola, or by the branches of the conjugate hyperbola.

143. *Every diameter is bisected by the centre of the hyperbola.*



Let P be any point in the hyperbola.

Join PC, and produce it to P', making $CP' = CP$.

Then we have to prove that P' is a point on the curve.

Join SP, $S'P'$; SP', $S'P$.

Then, since the triangles SCP, $S'CP'$ are equal in all respects, (Eucl. I. 4.)

$\therefore SP = S'P'$.

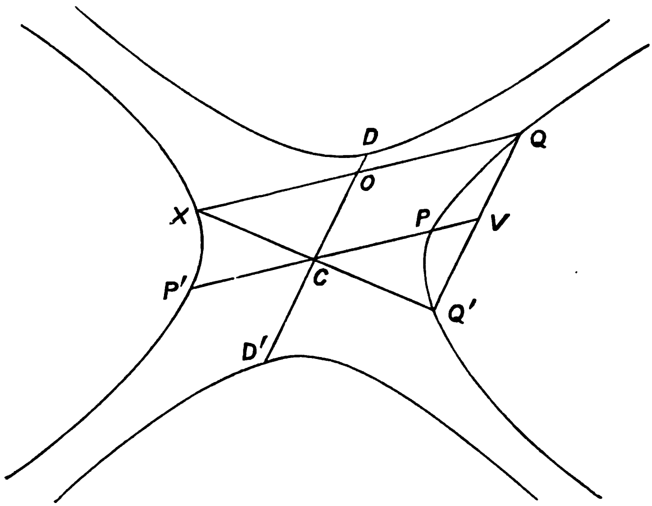
And since the triangles PCS', $P'CS$ are equal in all respects, (Eucl. I. 4.)

$\therefore S'P = SP'$.

Hence $SP' - S'P' = S'P - SP$
$= 2CA$;

$\therefore P'$ is a point in the curve;
and PP' is a diameter, bisected in C.

144. *If a diameter PCP' bisects chords parallel to another diameter DCD', then shall DCD' bisect chords parallel to PCP'.*



Let the diameter PCP' bisect the chords, as QQ', parallel to another diameter DCD'.

Then shall DCD' bisect the chords parallel to PCP'.

Let CP produced meet QQ' in V.

Join $Q'C$, and produce it to meet the other branch in X, and join XQ, cutting CD in O.

Then $Q'X$ is bisected in C, and QQ' is bisected in V;

$\therefore$ QX is parallel to CV, (Eucl. VI. 2.)

that is, QX is a chord parallel to PCP'.

Now, since CD is parallel to QQ',

and $CX = CQ'$,

$\therefore$ $OX = OQ$, (Eucl. VI. 2.)

that is, DCD' bisects any chord parallel to PCP'.

DEF.—Two diameters, PCP', DCD', each of which bisects the chords parallel to the other, are called *Conjugate Diameters*, and CP and CD are called *Semi-conjugate Diameters*.

Of any two conjugate diameters one meets the original hyperbola, and the other meets the conjugate hyperbola.

145. DEF. *Chords* of a hyperbola may be conveniently classed as *Internal* or *External*.

Internal Chords are straight lines joining two points in the same branch of the curve.

External Chords are straight lines joining two points in opposite branches of the curve.

Thus, in the figure of Art. 144, QQ' is an internal chord, and XQ is an external chord.

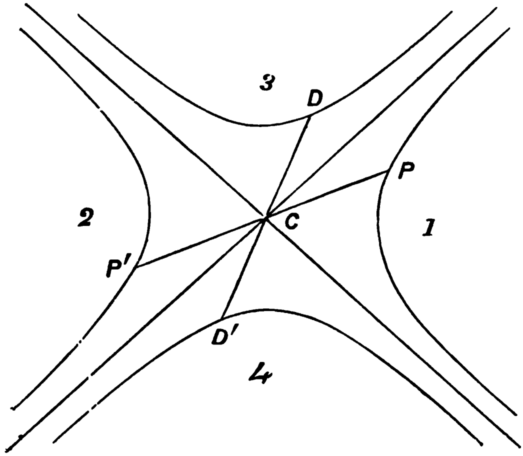
The Mutual Relations of the Conjugate Hyperbola and the Original Hyperbola.

146. The branches of the conjugate hyperbola must not be regarded as forming any part of the original hyperbola; but the four branches are mutually related: thus, taking PCP' and DCD' for conjugate diameters,

(1.) PCP' produced will bisect all chords in 1 and 2 parallel to DCD'; that is, all internal chords of the original hyperbola parallel to DCD'.

(2.) PCP', produced if necessary, will bisect all lines parallel to DCD', and terminated by 3 and 4; that is, all external chords of the conjugate hyperbola parallel to DCD'.

(3.) DCD' produced will bisect all chords in 3 and 4 parallel to PCP'; that is, all internal chords of the conjugate hyperbola parallel to PCP'.

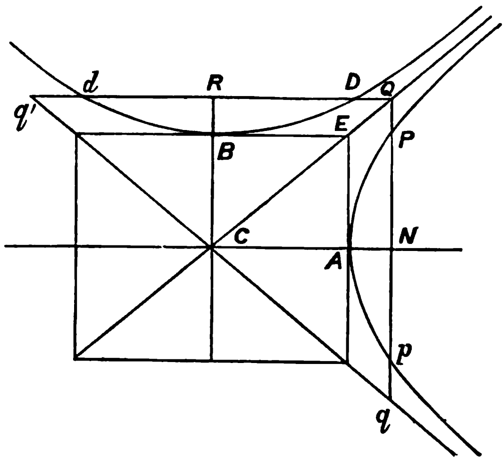


(4.) DCD', produced if necessary, will bisect all lines parallel to PCP' and terminated by 1 and 2; that is, all external chords of the original hyperbola parallel to PCP'.

(5.) The tangents at P and P' are parallel to DCD', and the tangents at D and D' are parallel to PCP'.

(6.) The portions of the tangents at P, P', D, D', intercepted by the asymptotes are bisected at P, P', D, D'.

147. *The properties of the conjugate hyperbola are analogous to those of the original hyperbola.*



From Q, a point in one of the asymptotes, draw Qq, Qq' perpendiculars to the axes, and meeting the original hyperbola and its conjugate in the points P, p, D, d, and the other asymptote in q, q'.

Now we have shown in Art. 137 that $QN^2 - PN^2 = CB^2$, and we shall now show that a similar relation holds good for the conjugate hyperbola; that is,

$$QR^2 - DR^2 = CA^2.$$

For $QR^2 : CR^2 = EB^2 : CB^2$ (Eucl. VI. 2.)

$= CA^2 : CB^2$

$= DR^2 : CR^2 - CB^2$; (Art. 107.)

$\therefore\ QR^2 : DR^2 = CR^2 : CR^2 - CB^2$;

$\therefore\ QR^2 - DR^2 : DR^2 = CB^2 : CR^2 - CB^2$;

$\therefore\ QR^2 - DR^2 : CB^2 = DR^2 : CR^2 - CB^2$

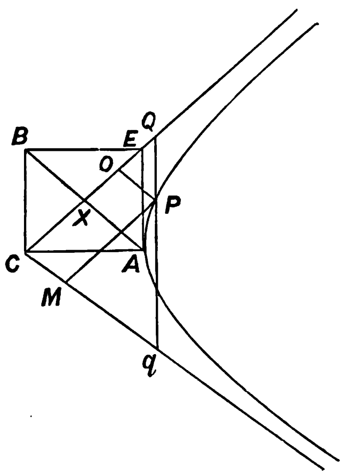
$= CA^2 : CB^2$; (Art. 107.)

$$\therefore\ QR^2 - DR^2 = CA^2.$$

$$\text{Hence } QD \,.\, Dq' = CA^2.$$

Hence also it follows, by a proof similar to that of Art. 140, 141, that $QD = q'd$.

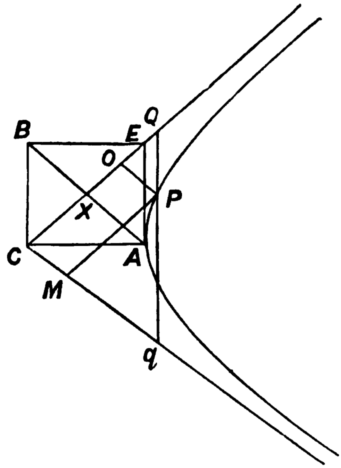
148. *If from any point on the curve, lines are drawn parallel to each asymptote and terminated by the other asymptote, the rectangle contained by these lines is constant, and equal to* $\frac{1}{4}(CA^2 + CB^2)$.



Let PO, PM be the lines parallel to the asymptotes meeting the asymptotes in O and M.

Draw QPq perpendicular to the axis.

Then, by similar triangles,

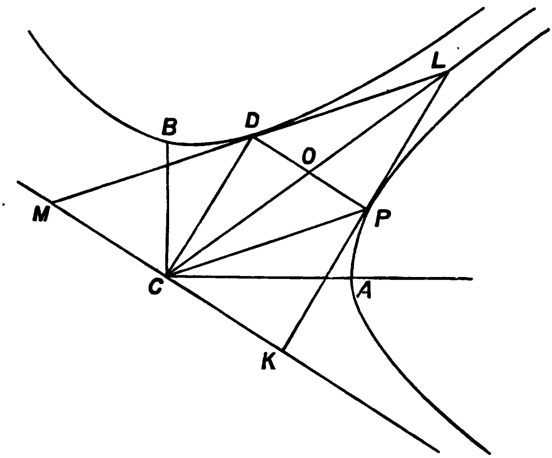


$$PO : PQ = AX : AE, \text{ and}$$
$$PM : Pq = XE : AE;$$
$$\therefore PO.PM : PQ.Pq = AX.XE : AE^2;$$
$$\therefore PO.PM : PQ.Pq = AX^2 : CB^2. \text{ (Eucl. I. 34, Ex. 1, 2.)}$$
$$\text{Now } PQ.Pq = CB^2; \qquad \text{(Art. 137.)}$$
$$\therefore PO.PM = AX^2$$
$$= \frac{1}{4}AB^2 \qquad \text{(Art. 8. 5.)}$$
$$= \frac{1}{4}(CA^2 + CB^2) \qquad \text{(Eucl. I. 47.)}$$
$$= \text{a constant quantity.}$$

Hence it follows that $PO.OC = \frac{1}{4}(CA^2 + CB^2)$.

A similar relation is true for the conjugate hyperbola.

149. *If from any point L on one of the asymptotes, tangents are drawn to the hyperbola and the conjugate hyperbola at the points P and D, the straight line PD is parallel to one asymptote and is bisected by the other, and CP, CD are conjugate semi-diameters.*



Let PD cut the asymptote CL in O. Let the tangents at P and D meet the other asymptote in K and M.

Then, since LK, LM are bisected in P, D, (Art. 142.)

$\therefore$ PD is parallel to MK; and $\therefore$ $LO = CO$.

Now $PO.OC = \frac{1}{4}(CA^2 + CB^2)$; (Art. 148.)

and $DO.OC = \frac{1}{4}(CA^2 + CB^2)$; (Art. 148.)

$\therefore$ $PO.OC = DO.OC$; and $\therefore$ $PO = DO$.

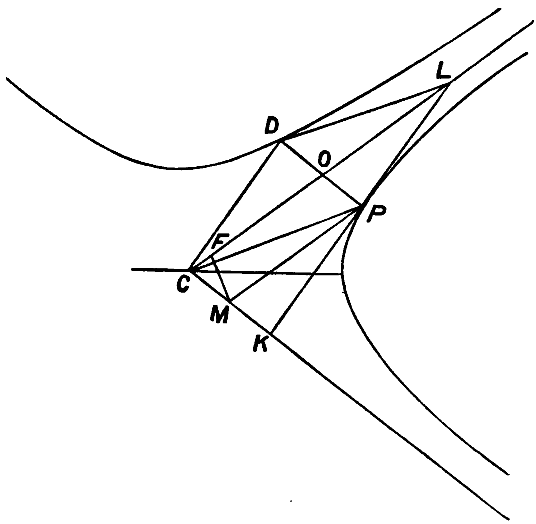
Then, since O is the middle point of LC and PD,

$\therefore$ $LPCD$ is a parallelogram;

and since CP bisects LK, which is parallel to CD,

$\therefore$ CP, CD are conjugate semi-diameters.

150. *If a tangent to the hyperbola at the point P meets the asymptotes in L and K, the triangle LCK and the parallelogram LDCP are of constant magnitude, and each is equal to CA . CB.*



Draw PM parallel to LC, and MF perpendicular to CO.

Then, since LCK is an angle of constant magnitude, the angle FCM is constant, and therefore, as MFC is a right angle, the angle FMC is of constant magnitude.

Hence, though the absolute values of MC and MF will vary according to the position of the point L, yet their ratio will always be the same.

Then, since $PO \,.\, PM =$ a constant magnitude, (Art. 148.)

$\therefore$ $MC \,.\, PM =$ a constant magnitude,

$\therefore$ $MF \,.\, PM =$ a constant magnitude,

$\therefore$ the parallelogram MO is of constant magnitude.

Also M is the middle point of CK,

because P is the middle point of LK.

Now triangle $LCK = 2$ triangle CPK

$= 4$ triangle CMP

$= 2$ parallelogram MO;

$\therefore$ triangle LCK is of constant magnitude.

Also the parallelogram $LDCP = 2$ triangle LCP

$=$ triangle LCK

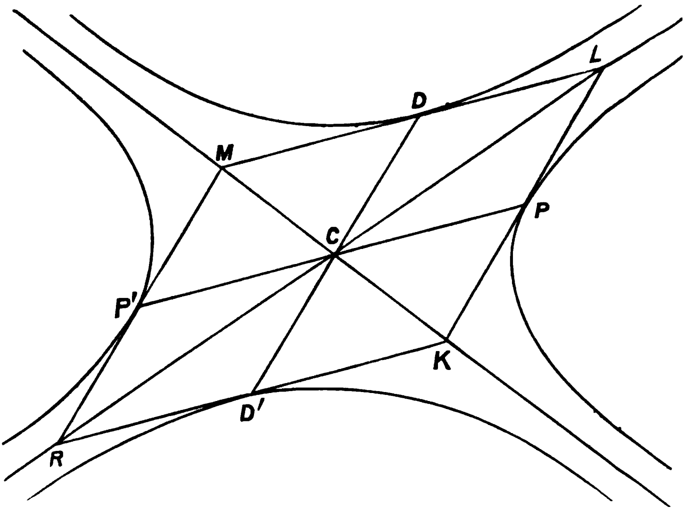
$=$ a constant magnitude.

Now when P coincides with A, the vertex of the hyperbola, D will coincide with B, the vertex of the conjugate hyperbola, and the parallelogram $LDCP$ becomes the rectangle contained by CA, CB.

Hence, in all positions of P,

area of parallelogram $LDCP = CA \,.\, CB$.

151. *The parallelogram formed by drawing tangents at the ends of two conjugate diameters, is constant in area and is equal to the rectangle $AA' \cdot BB'$.*



We learn from Art. 149 that the tangents at P and D, the ends of the conjugate diameters PP', DD', meet in the same point L on one of the asymptotes. Hence if M and K be the points in the other asymptote where it is cut by the tangents at P and D, it is evident that the tangents at P' and D', which meet at the point R on the asymptote LCR, will cut the other asymptote in M and K, and that $LKRM$ is a parallelogram.

Now $LKRM$ is made up of four equal parallelograms,

$$LC, KC, RC, MC,$$

since P, D', P', D are the middle points of

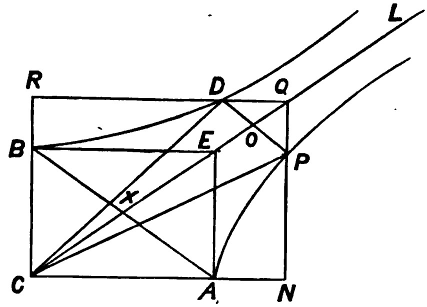
$$LK, KR, RM, ML.$$

$$\begin{aligned} \text{Hence area of } LKRM &= 4 \text{ area } LC \\ &= 4CA \,.\, CB \quad \text{(Art. 150.)} \\ &= AA' \,.\, BB'. \end{aligned}$$

COR. If PF be the perpendicular from P on the conjugate diameter,

$$\begin{aligned} PF \,.\, CD &= \text{area } LC \\ &= CA \,.\, CB. \end{aligned}$$

152. *If CP and CD be conjugate semi-diameters, the difference between CP^2 and CD^2 is equal to the difference between CA^2 and CB^2.*



Draw the ordinates PN, DR to the axes, and let NP produced meet the asymptote CL in Q.

Then, by similar triangles, POQ, AXE,

$$PO : OQ = AX : XE;$$

$$\therefore PO = OQ.$$

Now if RD be produced, it meets the asymptote for the same reason at a point such that

$$\begin{aligned} \text{distance of the point from } O &= OD \\ &= OP \\ &= OQ, \end{aligned}$$

that is, it meets the asymptote in Q.

Then $CQ^2 = QN^2 + CN^2$,

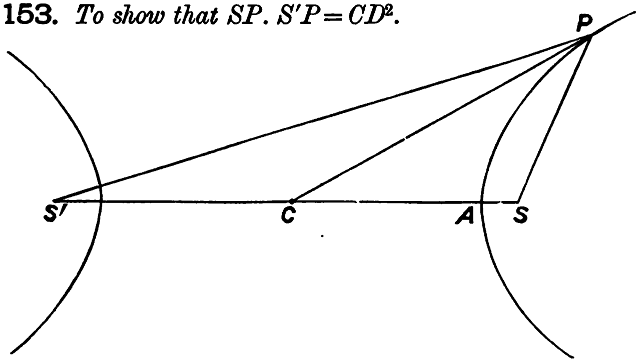
and $CP^2 = PN^2 + CN^2$;

$$\begin{aligned} \therefore CQ^2 - CP^2 &= QN^2 - PN^2 \\ &= CB^2; \qquad \text{(Art. 137.)} \end{aligned}$$

and $\therefore CQ^2 = CP^2 + CB^2$.

Similarly $CQ^2 - CD^2 = QR^2 - DR^2$
$= CA^2$; (Art. 147.)
and $\therefore CQ^2 = CD^2 + CA^2$.
Hence $CP^2 + CB^2 = CD^2 + CA^2$;
and $\therefore$ the difference between CP^2 and CD^2 is equal to the difference between CA^2 and CB^2.

153. *To show that $SP \cdot S'P = CD^2$.*



Since C is the middle point of SS',

(1.) $S'P^2 + SP^2 = 2CP^2 + 2CS^2$. (Art. 8 (8).)

Also, since $S'P - SP = 2CA$,

(2.) $\therefore S'P^2 + SP^2 - 2SP \cdot S'P = 4CA^2$. (Art. 9.)

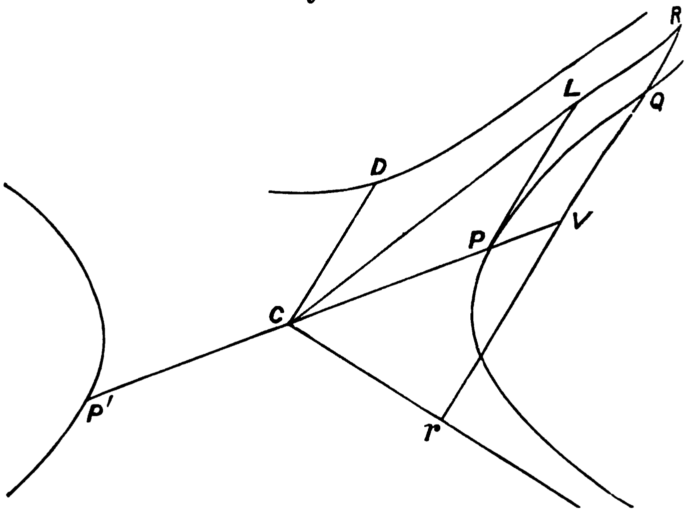
Subtracting (2) from (1),

$$2SP \cdot S'P = 2CP^2 + 2CS^2 - 4CA^2;$$

$$\therefore SP \cdot S'P = CP^2 + CS^2 - 2CA^2$$
$$= CP^2 + CS^2 - CA^2 - CA^2$$
$$= CP^2 + CB^2 - CA^2 \quad \text{(Art. 115.)}$$
$$= CD^2 + CA^2 - CA^2 \quad \text{(Art. 152.)}$$
$$= CD^2.$$

154. *If QV is an ordinate of the diameter PCP', and CP. CD are conjugate semi-diameters, then*

$$PV.VP' : QV^2 = CP^2 : CD^2.$$



Let QV produced meet the asymptotes in R, r, and draw PL a tangent at P to meet the asymptote in L,

Then $PL = CD$. (Art. 149.)

Now $RQ.Qr + QV^2 = RV^2$; (Eucl. II. 5.)

and $RQ.Qr = PL^2$; (Art. 142.)

$\therefore QV^2 = RV^2 - PL^2$.

Again, by similar triangles, CVR, CPL,

$CV^2 : CP^2 = RV^2 : PL^2$;

$\therefore CV^2 - CP^2 : CP^2 = RV^2 - PL^2 : PL^2$;

$\therefore CV^2 - CP^2 : CP^2 = QV^2 : CD^2$.

Now $CV^2 - CP^2 = PV.VP'$; (Eucl. II. 6.)

$\therefore PV.VP' : CP^2 = QV^2 : CD^2$,

or, $PV.VP' : QV^2 = CP^2 : CD^2$.

EXERCISES

EXERCISES

I.—The Parabola.

1. The diameter of the circle passing through the vertex and the ends of the latus rectum is equal to $5AS$.
2. If PSp, QSq be focal chords, prove that PQ, pq meet on the directrix.
3. The angle between the tangents at P and Q, two points on the curve, is equal to half the angle PSQ.
4. If tangents at P and Q, two points on the curve, intersect in R, then, whatever be the position of QR, the angle QRS is constant.
5. If AN be the abscissa of P, and Y the point where the tangent at P intersects the tangent at A, show that $PY^2=AN.SP$. Show also that, if YK be the perpendicular from Y to SP, $PN=2YK$.
6. A circle has its centre in the vertex A of a parabola whose focus is S; the diameter of the circle is $3AS$. Show that the common chord of the parabola and circle bisects AS.
7. If a circle be described about the triangle SPN, the tangent drawn to it from A will be equal to half PN.
8. If PG, the normal at P, meet the axis in G, and if the triangle SPG is equilateral, prove that SP is equal to the latus rectum.

9. A circle is described on the latus rectum as diameter, and a common tangent QP is drawn to it and to the parabola: show that SP, SQ make equal angles with the latus rectum.

10. If PC be drawn perpendicular to the chord AP to meet the axis in C, and PN be the ordinate of P, then NC is equal to the latus rectum.

11. If from any point T on a tangent to a parabola at T, TN, TR be drawn perpendicular to the focal distance SP and the directrix respectively, show that $SN=TR$.

12. If the tangent at P meet the axis in T and the directrix in D, and if the angle PST be equal to two-thirds of a right angle, show that $PT=TD$.

13. If P, Q be two points on a parabola such that PAQ is a right angle, prove that PQ cuts the axis at a point whose distance from the vertex is equal to the latus rectum.

14. The normal chord to a parabola at the point whose ordinate is equal to its abscissa subtends a right angle at the focus.

15. If PSQ be a focal chord, show that $SP^2=PQ\,.\,AN$.

16. Show that the tangent at any point of the parabola meets the directrix and the latus rectum produced in points equally distant from the focus.

17. If from G, the foot of the normal at P, GK be drawn perpendicular to SP, show that $PK=2AS$.

18. If SD be drawn from S perpendicular to the normal at P, and if PN be the ordinate of P, show that $SD^2=SP\,.\,AN$.

19. If PSQ be a focal chord, and SL be the semi-latus rectum, show that $2SP\,.\,SQ=SL\,.\,PQ$.

20. If the tangents at the extremities of the focal chord PSQ meet the tangent at the vertex in the points Y and Z, show that $YZ^2 = AS . PQ$.

21. If the normal at P, produced to meet the curve again at Q, subtend a right angle at the focus, the ordinate of P is equal to the latus rectum.

22. If the normal at the extremity L of the latus rectum intersect the parabola in P, and the ordinate PN be drawn, show that $PN = 3SL$.

23. PSQ is a focal chord. Prove that AP, AQ will meet the latus rectum in two points whose distances from the focus are equal to the ordinates of P, Q respectively.

24. If the normal at P meets the axis in G, and the ordinate at G meets the parabola in Q and AP produced in R, then $QG^2 = PN . RG$.

25. If from P two lines PF, PH be drawn making equal angles with the normal PG, then $SG^2 = SF . SH$.

26. If PQ be any chord, PN, QM ordinates at its extremities, and O the point where it cuts the axis, then $AN . AM = AO^2$.

27. A parabola being traced, and its axis given, determine the focus.

28. If P be a point in a parabola, show that the circle described on SP as diameter touches the tangent at the vertex.

29. Show that the directrix is the locus of the points from which tangents at right angles to each other can be drawn to the parabola.

30. If PYT be a tangent meeting the tangent at the vertex in Y, and the directrix in D, then $PY . PD = SP^2$ and $PY . YD = AS . SP$.

31. If the tangents at P and Q intersect in R, and PM, QM', RN be drawn perpendicular to the axis, then $2RN=PM+QM'$ or $2RN=PM-QM'$, according as P and Q are on the same or different sides of the axis.

32. If PM be drawn perpendicular to the chord AP, to meet the axis in M, and if PN be the ordinate of P, then $MN=$ the latus rectum.

33. If PG be the normal at P, and PN the ordinate of P, show that $AP^2=AN(AN+2NG)$.

34. If from any point P in the parabola the straight lines PF, PH be drawn to the axis, making equal angles with the normal PG, then $SG^2=SF.SH$.

35. If the tangent and normal at P meet the axis in T and G, and PN be the ordinate of P, show that $PT^2=4AN.SP$ and $PG^2=4AS.SP$.

36. If a right-angled triangle be inscribed in a parabola, having its base perpendicular to the axis and its vertex at the vertex of the parabola, its area equals the square on the latus rectum.

37. If from the intersection of any two tangents a straight line be drawn to the focus, and another straight line be drawn perpendicular to the directrix, they will make equal angles with the tangents.

38. If PT, QT be tangents at P and Q, then $PT^2 : QT^2=SP : SQ$.

39. If PT, pT be tangents to a parabola, and they be cut by a third tangent in Q and R, then $TQ : pR=QP : RT$.

40. If two equal tangents be drawn to a parabola, the alternate segments of them made by a third tangent will be equal.

41. If two tangents be drawn at the extremities of a chord, and a third tangent be drawn parallel to the chord, the portion of this, which is intercepted by the other tangents, is bisected at the point of contact.

42. S is the focus and PSq a focal chord in a parabola: show how to construct the curve.

43. A is the vertex of a parabola, and AP a chord inclined to the axis at an angle of 60°. Show that the abscissa of P is one-third of the latus rectum, and that the perimeter of the triangle APS is three times half the latus rectum.

44. If the portion of the normal intercepted between the directrix and the point P, at which the normal is drawn, be equal to twice the latus rectum, find the focal distance of the point P.

45. P and Q are two points on a parabola whose vertex is A. If AQ be perpendicular to the tangent at P, then AP is perpendicular to the tangent at Q.

46. Two tangents are drawn to a parabola from a point on the directrix. Show that the segment which they intersect on the tangent at the vertex is equal to the distance of the given point from the focus.

47. If in the parabola $PY = PG$, then $AN = 4AS$, and if a circle be described about the triangle YNG, it will cut off from the tangent at the vertex a portion equal to TG.

48. If TP, TQ be tangents to a parabola, and T be a point in the directrix, prove that the circle circumscribing the triangle TPQ touches the directrix.

49. If the ordinate of P bisects the subnormal of P', prove that the ordinate of P is equal to the normal of P'.

50. From the vertex of a parabola any two chords are drawn at right angles to each other. Prove that the tangents at their extremities intersect on a fixed straight line parallel to the directrix.

51. From a point O are drawn two tangents to a parabola, and on these tangents are taken P and Q, such that $SP=SQ=SO$. Prove that PQ will be perpendicular to the axis.

52. If AN, AN' be the abscissæ of the extremities of a focal chord, and G, G' be the points of intersection of the normals at the extremities with the axis, show that $AN : AN' = SG^2 : SG'^2$.

53. Two tangents OP, OQ are drawn to a parabola. The tangents at P, Q to the circle OPQ meet in R. Prove that RO produced will pass through S.

54. A chord PQ of a parabola is normal at P, and is bisected by the diameter drawn from P'. Prove that PP' passes through the focus.

55. The normal at P meets the axis in G, and the chord QGQ' is drawn perpendicular to the axis; on this chord a point R is taken such that $GR=GP$. Prove that the rectangle $QR \, . \, RQ'$ is equal to the square on the semi-latus rectum.

56. If N be any point in a double ordinate PP' perpendicular to the axis of a parabola, whose latus rectum is $4AS$, and if NQ be drawn parallel to the axis to meet the parabola in Q, prove that $PN \, . \, NP' = 4AS \, . \, QN$.

57. P is a point in a parabola such that, if PG is the normal, the triangle SPG has each of the angles at the base double the vertical angle. If the angle SGP be bisected by a straight line GK cutting SP in K, prove that PK is equal to the latus rectum.

58. A circle, which touches a parabola in P and cuts it in Q and R, also passes through the focus and cuts the axis in O. Prove that $PQ=RO$.

59. PP' is a chord of a parabola normal at P, and the diameter bisecting it meets the curve in Q; the chord QQ' is drawn normal at Q. Prove that QQ' is bisected by the diameter through P.

60. TP, TQ are two tangents to a parabola. Prove that the perpendiculars let fall from P, T, Q, or any other tangent to the parabola, are in geometrical progression.

61. P is any point in a parabola whose vertex is A; the focal chord QSQ' is parallel to AP; PN, QM, $Q'M'$ being perpendiculars to the axis, show that SM is a mean proportional between AM and AN, and that $MM'=AP$.

62. PSQ is a focal chord, and on PS and QS, as diameters, circles are described. Prove that the length of either of their common tangents is a mean proportional between AS and PQ.

63. If AQ be a chord of a parabola through the vertex A, and QR be drawn perpendicular to AQ to meet the axis in R, prove that AR is equal to the chord through the focus parallel to AQ.

64. G is the foot of the normal at P, Q is the middle point of SG, and X is the foot of the directrix. Show that $QX^2-QP^2=4AS^2$.

65. Show that the point of intersection of the tangents at any two points on a parabola is equidistant from the diameters through those points.

66. TP, TQ are tangents from T, and TS meets PQ in R; RZ, parallel to the axis, meets the directrix in Z. Show that PZ and QZ are equally inclined to the axis.

67. PSQ is a focal chord and V its middle point. Show that $PV^2=AV^2+3AS^2$.

68. PQ is any chord, ABU any diameter meeting the tangent at P, the curve, and the chord in the points A, B, U respectively. Prove that $AB:BU=PU:UQ$.

69. If P be a point in a parabola, and a circle be described touching the axis of the parabola and the focal distance SP in P, then the portion of the diameter through P which is cut off by this circle is equal to the semilatus rectum.

70. PN is an ordinate of a point P in a parabola, QR is a diameter bisecting PN, and cutting the curve in Q; NQ cuts the tangent at the vertex A in T. Show that $3AT=2PN$.

71. If a circle be described touching a parabola in P and its axis in the focus, then the normal at P will make with the axis an angle equal to one-third of a right angle.

72. A circle is described, touching the axis at the point where it meets the directrix, and a line perpendicular to the axis, touching the circle in R, meets the parabola in P and Q, and the axis in N. Show that $SN^2=PR\,.\,QR$.

73. A normal is drawn to a parabola at a point whose abscissa and ordinate are each equal to the latus rectum; show that it meets the parabola at a point whose distance from the axis is three times the semilatus rectum.

74. Through any point P of a parabola a straight line QPQ' is drawn parallel to the latus rectum and is terminated by the tangents at the extremities of the latus rectum. PR is drawn perpendicular to the latus rectum, produced if necessary. Show that $PR^2=QP\,.\,PQ'$.

75. If the normal PG meets the directrix in F, show that PF varies directly as PS^2.

76. If the normal PG meets the directrix in F, and the perpendicular from S on the normal meets the normal in K, show that $FK\,.\,AS=SY\,.\,AG$.

77. If QV be the semi-ordinate to the diameter PV, and QF be perpendicular to the ordinate PR, and if QR and VF be joined, show that the triangles PVF, FQR are similar.

78. PSQ is a focal chord of a parabola whose vertex is A. If through P and Q lines be drawn perpendicular to the axis, and meeting AQ, AP in p and q, show that $Pp \cdot Qq = 4AS \cdot PQ$.

II.—The Ellipse.

1. Show that the axis major is greater than any other diameter.
2. Show that the axis minor is less than any other diameter.
3. If $CA = 2CB$, find the numerical value of the eccentricity.
4. Show that the tangent at any point makes a greater angle with the focal distance than with the perpendicular on the directrix.
5. Show that the tangent at the extremity of the latus rectum intersects the axis major in the directrix.
6. If any ordinate NP be produced to meet the tangent at the extremity of the latus rectum in Q, prove that $QN = SP$.
7. If PSQ be a focal chord, and X the point where the directrix meets the axis, show that XP, XQ, are equally inclined to the axis.
8. If S be a focus, T any point on the tangent at P, and TL, TM, perpendiculars from T upon SP and the directrix, show that $SL : TM = SA : AX$.
9. If P be any point in the ellipse, and the chords AP, $A'P$, be produced to intersect a perpendicular to the axis through T (the point where the tangent at P meets the axis major) in Q and Q', then $QT = Q'T$.

10. If X be the point where the directrix meets the axis major produced, show that $CX : SX = CA^2 : CB^2$.

11. If AP, $A'P$ be chords drawn from the vertices to a point P in the ellipse, and PK, PK' be perpendiculars to them, the line KK' intercepted on the axis is equal to the latus rectum.

12. Prove that when the angle SBS' is a right angle $CA^2 : CB^2 = 2 : 1$.

13. If an ellipse and a parabola have the same vertex A and the same focus S, and if A' be the other extremity of the major axis of the ellipse, then $2AC : A'S =$ latus rectum of parabola : latus rectum of ellipse.

14. If a circle be described through the foci of an ellipse, a straight line drawn from its intersection with the minor axis to its intersection with the ellipse will be a tangent to the ellipse.

15. SL is the semi-latus rectum, A the vertex, LA produced meets the directrix in Q, and QS intersects the tangent at the vertex in R; prove that $AR = AS$.

16. If the focal distances of a point P on the ellipse be perpendicular, show that the distance of P from either focus is double that of the tangent at P.

17. If the tangent at P meet the tangent at the vertex A in T, and S' be the focus furthest from A, then TA is equal to the perpendicular from T on $S'P$.

18. Tangents drawn to an ellipse from any point in the circumference of the circle described on the axis major include an obtuse angle.

19. If P be a point on an ellipse such that the angle subtended at P by the line joining the foci is a right angle, show that the ordinate of P is equal to the distance of a focus from the corresponding directrix.

20. If any number of ellipses be described on the same major axis, tangents at the extremities of the latera recta all pass through the same point.

21. If Y and Y' be the feet of the perpendiculars from S and S' on the tangent at P, prove that CY bisects SP.

22. If Y and Y' be the feet of the perpendiculars from the foci on the tangent at P, and if the circles described on SY, $S'Y'$ as diameters cut SP, $S'P$ in K and L respectively, then KL will be parallel to the major axis.

23. If any number of ellipses be described with the same minor axis, and be cut by a common ordinate to that axis, the tangents at the points where the common ordinate cuts the curve will all pass through the same point.

24. PN, PT are an ordinate and tangent at P. Show that the circles, whose diameters are NT and the major axis, intersect at right angles.

25. If the tangent at the end of the latus rectum meet the major and minor axes in T and t, and a circle be drawn about the triangle STt, this circle will touch the minor axis in t.

26. If the tangent at P meets the major axis in T, and Y and Y' be the feet of the perpendiculars from the foci on the tangent at P, show that $TY \,.\, TY' - PY \,.\, PY' = PT^2$.

27. If the ordinate at P meet the auxiliary circle in Q, and the tangent to the circle at Q meet the axis minor produced in R, then $RC \,.\, PN = CA \,.\, CB$.

28. The tangent at P meets the axes in T, t. Join St, and prove that angle $PSt =$ angle STP.

29. PP' is a diameter, $P'S$ and $P'S'$ meet the tangent at P in Y' and Z'. Prove that the sum of $P'Y'$ and $P'Z'$ is twice the major axis.

30. If NP be the ordinate of a point P, and if Y and Y' be the points where the tangent at P meets the perpendiculars from the foci, show that $NY : NY' = PY : PY'$.

31. If from the centre of an ellipse lines be drawn parallel and perpendicular to the tangent at any point, they will enclose a part of one of the focal distances of that point, which shall be equal to the other focal distance.

32. If from S' a line be drawn parallel to SP, it will meet SY, the perpendicular on the tangent at P, in the circumference of a circle.

33. The external angle between any two tangents to an ellipse is equal to the semi-sum of the angles which the chord joining the points of contact subtends at the foci.

34. If AQ be drawn from one of the vertices perpendicular to the tangent at any point P, prove that the locus of the point of intersection of PS and QA produced will be a circle.

35. If a circle passing through Y and Y', the feet of the perpendiculars from the foci on the tangent at P, touch the major axis in Q, and if that diameter of the circle which passes through Q meet the tangent in P, then $PQ = CB$.

36. If Y and Y' be the feet of the perpendiculars from the foci on the tangent at P, prove that the circle circumscribed about the triangle YNY' (PN being the ordinate of P) will pass through C.

37. From a point O in the auxiliary circle tangents are drawn to the ellipse, touching it in P and Q, and meeting the auxiliary circle again in p and q. Show that the angle pCq is equal to the sum of the angles PSQ and $PS'Q$.

38. Tangents are drawn from any point on the auxiliary circle to the ellipse. Prove that the line joining one of the points of contact with one of the foci is parallel to the line joining the other point of contact with the other focus.

39. Tangents are drawn from a point T to the ellipse. The chord of contact and the major axis, or these produced, intersect in K, and TN is drawn perpendicular to the major axis. Prove that $CN.CK=CA^2$.

40. Tangents are drawn to an ellipse from any point on the line through the focus perpendicular to the axis: prove that the length intercepted by them on the corresponding directrix is bisected by the axis.

41. If a circle be described touching SP, $S'P$ produced, and the major axis of an ellipse (P being a point on the curve), find the locus of the centre of the circle.

42. Perpendiculars from S and C meet the tangent at P in Y and Z, and XY meets the minor axis in T. Prove that if $CP=CS$, then $CZ=ZT$.

43. If a tangent at a point P on an ellipse meet the tangents at the vertices in H and K and the minor axis produced in Q, show that $HA : CQ=PN : A'K$.

44. If the tangent at P meet the tangents at the vertices in H and K, show that HK is a diameter of a circle which passes through the foci.

45. Y and Y' being the feet of the perpendiculars from the foci upon a tangent at P, show that the angle YAY' is equal to half the angle SPS'.

46. If a tangent at P meet the major axis in T, and the perpendiculars from S and C in Y and Z, then $TY^2 : PY^2=TZ : PZ$.

47. If the tangent at P meet the major axis in T, and the tangent at A meet CP produced in E, then the triangles ACE, PCT are equal.

48. A circle passes through the focus S, has its centre on the major axis, and touches the ellipse. Show that the straight line from S to the point of contact is equal to the latus rectum.

49. SP is the focal distance of a point in an ellipse; CR is a radius of the auxiliary circle parallel to SP, and is drawn in the direction from P to S; SQ is perpendicular to CR. Show that $SP \, . \, QR = CB^2$.

50. Perpendiculars SY, $S'Y'$ are drawn from the foci on a pair of tangents TY, TY'. Prove that the angles STY, $S'TY'$ are equal or supplementary to the angles at the base of the triangle formed by joining Y and Y' to the centre of the ellipse.

51. If Q be a point on the major axis, and if P be a point on the ellipse such that $CP = BQ$, prove that $AQ = SP$ and $A'Q = S'P$.

52. A circle is described passing through the foci, and common tangents are drawn to the ellipse and the circle. If the points of contact on the circle on the same side of the major axis be joined, the line so drawn will pass through the extremity of the minor axis.

53. If from the ends of a diameter of the auxiliary circle two pairs of tangents be drawn to an ellipse, show that their other points of intersection lie on the directrix.

54. SY and CZ are the perpendiculars from the focus and the centre upon the tangent at any point P. Show that $SY : CZ = SP : CA$.

55. Tangents are drawn to the ellipse from a point O. If on the tangents be taken points Q, R such that $SQ = SR = SO$, then will $S'O$ be perpendicular to QR.

56. If P be any point on the curve and with centres S, S' and radii SP, $S'P$ two circles be described, their common tangents will touch the auxiliary circle.

57. If the normal PG cut the minor axis in g, prove that PSg and $PS'g$ are right angles.

58. PSQ is a focal chord. The normals at P and Q intersect in K, and KN is drawn perpendicular to PQ. Prove that $PN=SQ$.

59. From the foci S, S', SO and $S'O'$ are drawn perpendicular to SP and $S'P$, to meet the normal at P in O and O'. Show that OO' is bisected by the minor axis.

60. If the normal at P pass through an extremity of the minor axis, then the circle described on SS' as diameter will touch the tangent at P.

61. PG is a normal, terminating in the major axis. The circle, of which PG is a diameter, cuts SP, $S'P$ in K, L respectively. Prove that KL is bisected by PG and is perpendicular to it.

62. The normal at P meets the tangent at the other end of the focal chord through P in R; from R a perpendicular RK on that chord is drawn. Show that PK is equal to the major axis.

63. The tangent to an ellipse at P meets the major axis in T; the ordinate at P meets the auxiliary circle in Q; the normal at P meets the major axis in G. Show that $TQ : TP = CB : PG$.

64. If the normal at the extremity of the latus rectum cut the minor axis in R, show that CR is a third proportional to SA and CS.

65. Through a point P on an ellipse a line PDE is drawn cutting the axes so that the segments PD and PE are equal to the two semi-axes respectively: perpendiculars to the axes through D and E intersect in O. Prove that OP is a normal.

66. If from G, the foot of the normal at P, GR be drawn perpendicular to SP, show that $2PR=$the latus rectum.

67. If G be the foot of the normal at P, and SY the perpendicular on the tangent at P, show that $SP : SY = 2PG :$ the latus rectum.

68. If the normal at P cut the minor axis in g, show that g, S and S' lie on the circumference of a circle whose centre is on BC.

69. The normals at the ends of a focal chord meet in W, and the corresponding tangents meet in Z. Show that ZW passes through the other focus.

70. The tangents at P and D, the extremities of two conjugate diameters, meet CA, CB produced in T and R, prove that TR is parallel to AB.

71. CP and CD are conjugate semi-diameters. PQ is a chord parallel to one of the axes. Show that DQ is parallel to one of the straight lines joining the ends of the axes.

72. If through A a line be drawn parallel to a semi-diameter CP, meeting the conjugate diameter in V; and if PN be the ordinate of P, prove that the triangles ACV and CPN are equal.

73. If the diameter conjugate to CP meet SP and $S'P$, or these produced, in E and F, prove that $SE = S'F$.

74. The tangent at any point P is cut by any two conjugate diameters in T, t, and the points T, t are joined with the foci S, S' respectively. Prove that the triangles SPT, $S'Pt$ are similar.

75. If PG, the normal at P, cut the major axis in G, and if DR, PN be the ordinates of D and P, the extremities of conjugate diameters, prove that the triangles PGN, DRC are similar.

76. A circle is described on the minor axis; CP, CD are conjugate semi-diameters. Show that the tangent drawn from D to the circle is parallel to SP.

77. The normals at P and D, the extremities of conjugate semi-diameters, intersect the major and minor axes in G, K, G', K'. Prove that the triangles CGK, $CG'K'$ are equal.

78. Two tangents TP, TQ are drawn to an ellipse $PqpQ$, whose centre is C; Cp, Cq are the respectively parallel semi-diameters; Tp, PC are produced to meet in L, and Tq, QC to meet in M. Prove that the triangles TLP, TMQ are equal.

79. If from B, the extremity of the minor axis, two straight lines BK, BL be drawn parallel to CP, CD, two conjugate semi-diameters, and meeting the major axis in K, L, than $CK \cdot CL = CA^2$.

80. The ordinates at P and D, the extremities of two conjugate semi-diameters, meet the auxiliary circle in K and L. Show that the perpendiculars dropped from P and D on CK, CL respectively are equal.

81. Two conjugate semi-diameters CP, CD are produced to meet two parallel tangents in p, d respectively. Prove that pd touches the ellipse.

82. CP, CD are conjugate semi-diameters. From D tangents are drawn to the circle described on the minor axis as diameter. Prove that they are parallel to the focal distances of P.

83. Draw a pair of conjugate diameters containing a given angle.

84. If P and D be the extremities of conjugate diameters, show that $SP \cdot S'P + SD \cdot S'D = CA^2 + CB^2$.

85. For what position of the point P on an ellipse is the angle SPS' greatest?

86. When is the square of the sum of a pair of conjugate diameters the least?

87. If P and D be the extremities of conjugate diameters, prove that the rectangle contained by the abscissae of P and D is to the rectangle contained by the ordinates of P and D as the square on AA' is to the square on BB'.

88. CP and CD are conjugate semi-diameters. If CD cuts SP in E, then $SE^2+CD^2=CA^2$.

89. The radius of the circle described about the triangle SPS' is equal to $\frac{CD^2}{PN}$, PN being the ordinate of P, and CD the semi-diameter conjugate to CP.

90. If CP and CD be conjugate semi-diameters, and if Y be the foot of the perpendicular from the focus S on the tangent at P, show that

$$SY^2 : CB^2 = SP : S'P = SP^2 : CD^2.$$

91. If a chord AQ drawn from the vertex be produced to meet the minor axis in O, and CP be a semi-diameter parallel to the chord, then $AQ \,.\, AO = 2\,CP^2$.

92. TP is a tangent at P, and meets the major axis in T; a circle described round SPS' cuts TP in t; CD is conjugate to CP. Show that $CD^2 = TP \,.\, tP$.

93. Through any point P of an ellipse lines are drawn parallel to equal conjugate diameters to meet the major axis in K, L, and the minor axis in R, V.

Show that $KR^2+LV^2=2(CA^2+CB^2)$.

94. Let the tangents at P and Q, two points on an ellipse, meet in T, and let ST and $S'T$ cut PQ in K and L. Prove that $\frac{SQ}{KQ}+\frac{S'P}{LP}=4\,\frac{CA}{PQ}$.

95. Of all parallelograms circumscribing an ellipse those whose sides are parallel to conjugate diameters are the least.

96. A parallelogram circumscribes an ellipse. Show that the circles, each of which passes through the extremities of a side of the parallelogram and through the focus, are all equal.

97. If EF be one side of a parallelogram described about an ellipse, having its sides parallel to conjugate diameters, and the lines joining E, F to the foci intersect in O, O'; show that O, S, O', S' lie on a circle.

III.—The Hyperbola.

1. If S be a focus, T any point on the tangent at P, and TL, TM perpendiculars from T upon SP and the directrix, show that $SL : TM = SA : AX$.

2. Through N, the foot of the ordinate at P, draw NQ parallel to AP to meet CP in Q. Prove that AQ is parallel to the tangent at P.

3. If a point K be taken in the major axis, such that CK is a third proportional to CS and CA, and a perpendicular to the axis be drawn through K, the distance of any point in the curve from this line will bear a constant ratio to its distance from S.

4 If CP, CD be conjugate semi-diameters, and through C a straight line be drawn parallel to either focal distance of P, the perpendicular let fall from D on this straight line is equal to CB.

5. If CP, CD be conjugate semi-diameters, and DM, DN be perpendiculars on CA, show that

$$DM : CN = BC : CA = PN : CM.$$

6. Prove that a perpendicular drawn from a focus to an asymptote will intersect it in the directrix.

7. If the tangent at P meet one asymptote in T, and a line TQ be drawn parallel to the other asymptote to meet the curve in Q; prove that if PQ be joined and produced both ways to meet the asymptotes in R and R', RR' will be trisected in P and Q.

8. A circle is described with the focus as centre, and radius equal to one-fourth of the latus rectum. Prove that the focal distances of the points at which it intersects the hyperbola are parallel to the asymptotes.

9. From the point of intersection of an asymptote and a directrix of a hyperbola, a tangent is drawn to the curve. Prove that the line joining the point of contact with the focus is parallel to the asymptote.

10. If the normal at P meets the minor axis in g, then Pg will be to Sg in a constant ratio.

11. If P be any point such that the tangent at P meets an asymptote in T, the angle between that asymptote and $S'P$ is double the angle STP.

12. From a point K on the conjugate hyperbola the straight line $KQPpq$ is drawn to meet the hyperbola in P, p, and the asymptotes in Q, q.

Show that $KP \,.\, Kp = 2KQ \,.\, Kq$.

13. The tangent at a point P meets the transverse axis in T, and one of the asymptotes in L; PN, PG are the ordinate and normal at P.

Prove that $PG \,.\, PL : GT \,.\, CN = CB : CA$.

14. A tangent is drawn to a hyperbola at P and it meets one of the asymptotes in T. Prove that the triangles SPT, $S'PT$ are similar.

15. From the point in which the tangent at any point P cuts either asymptote, perpendiculars are drawn to the axes. Prove that the line joining the feet of these perpendiculars passes through P.

16. A straight line is drawn through the focus S parallel to one asymptote and meeting the other; prove that the part intercepted between the curve and the asymptote is one-fourth of the transverse axis, and the part between the curve and the focus one-fourth of the latus rectum.

17. If PN be the ordinate of a point P, NQ a tangent to the circle on the transverse axis, and PM parallel to QC, then $MN = CB$.

18. If from P, a point on the hyperbola, $RPQS$ be drawn, meeting the hyperbola in P and Q, and the asymptotes in R and S, then PK and QL being drawn parallel to one asymptote to meet the other, $LS = PK$.

19. If any two tangents be drawn to a hyperbola, and their intersections with the asymptotes be joined, the joining lines will be parallel.

20. If from a point P in the hyperbola PR be drawn parallel to an asymptote to meet the directrix in R, then $PR = SP$.

21. The part of the tangent to a hyperbola intercepted between the asymptotes is equal to the diameter at the point of contact.

22. If A and S be a vertex and focus of a hyperbola, and the tangent at the vertex and the directrix meet the asymptote in G and R respectively, then SG is parallel to AR.

23. In a rectangular hyperbola the eccentricity is the ratio of the diagonal of a square to the side of the square.

24. In a rectangular hyperbola, if a perpendicular from the focus meet the asymptote in B, then $CB = CA$.

25. If S and S' be the foci of a rectangular hyperbola, and a circle be described on SS', the tangents at the vertices will intersect the asymptotes in the circumference.

26. In a rectangular hyperbola CP is equal to the length of the normal at P.

27. If from a point Q in the conjugate axis of a rectangular hyperbola QA be drawn to the vertex, and QP parallel to the transverse axis to meet the curve, $QP = AQ$.

28. If from the point G, where the normal at P meets the axis, GR, GK be drawn to the points where the tangent at P intersects the asymptotes, the angle RGK is a right angle.

29. If from any point in a rectangular hyperbola lines be drawn to the extremities of a diameter, these lines will make equal angles with the asymptotes.

30. From G, the foot of the normal at any point P on a rectangular hyperbola, GE is drawn perpendicular to CP produced. Prove that PE is equal to the perpendicular from P on the conjugate semi-diameter CD.

31. In the rectangular hyperbola the perpendiculars drawn from the foci on the tangent at the extremity of the latus rectum are to one another as 3 : 1.

32. If the ordinate at any point P in a rectangular hyperbola meet the asymptote in Q, then CQ is an arithmetic mean between SP and $S'P$.

THE END.

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